Supplementary Online Material for “Self-organized Origami”

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Understanding the process of Miura-ori patterning requires a weakly nonlinear analysis of the straight dimensional periodic buckling pattern in the neighborhood of the primary instability that selects a preferred wavelength, but no preferred direction. In a large system, i.e. one where the wavelength of the primary pattern is small compared to the size of the system, tensile stresses along the buckles (or equivalently compression perpendicular to them) lead to the so-called Eckhaus instability, wherein a localized defect arises because it is easier for the buckles to respond to a weak expansion by localizing the deformation rather than accommodating it over the entire domain. On the other hand, compressive stresses along the buckles (or equivalently tension perpendicular to them) lead to the so-called zigzag instability which arises as a consequence of the fact that zigzag patterns allow for the accommodation of the effective lateral expansion of the buckles without changing the wavelength, but by simply tilting the folds; however since the folds can tilt in one of two equivalent directions, the most natural response of the system is to alternate the tilted regions to accommodate the lateral expansion at the expense of introducing localized defects. Our arguments so far are based on simple dimensional and geometric considerations. Indeed these instabilities of one-dimensional cellular patterns are independent of the detailed mechanisms at work and have been observed in other physical systems such as fluid convection, liquid crystals, superconductivity, flames etc. and reflect only the underlying symmetries of the system and the way in which they are broken (S1).

In the specific case of interest here, since the sheet is attached to a soft substrate, delamination does not occur and the deformations of the substrate are slaved to those of
the sheet. In the bulk the polymer substrate satisfies the equations of elastic equilibrium. Assuming that the polymer is isotropic and linearly elastic, with Young’s modulus $E_p$, Poisson ratio $\nu_p$ and depth $l$ we may then write (S2)

$$\Delta u_i + \frac{1}{1-2\nu_p} \partial_j (\partial_k u_k) = 0$$

(1)

where $u_i(x,y,z)$ are the displacements of the polymer. At the base of the polymer of thickness $l$, we assume that $u_i(x,y,-l) = 0$. At the interface with the thin stiff skin of thickness $h(<<l)$, Young’s modulus $E$, and Poisson ratio $\nu$, the continuity of traction and displacement implies that

$$Eh^2 \Delta \xi_{ij} - \partial_a (\sigma_{ajp} \partial_p \xi_a) = \frac{p_z}{h}$$

$$\partial_a \sigma_{ajp} = \frac{p_p}{h}$$

(2)

Here $u_i(x,y,0) = \xi(x,y)$ is the out-of-plane deformation, $\sigma_{ajp}(x,y)$ is the two-dimensional in-plane stress field in the plate, $h^2 = h^2 / 12(1-\nu^2)$, and $p_p, p_z$ are the in-plane and out-of-plane components of the tractions at the polymer-skin interface. In fact Eq. (2) are the Foppl-von Karman equations (S2) that are valid for the geometrically nonlinear deformations of a plate. Seeking linear periodic perturbation solutions to Eq. (1)-(2) of the form: $\xi(x,y) = A_k e^{i(k_xx+k_yy)} + A^*_k e^{-i(k_xx+k_yy)}$ where $A_k$ is the amplitude and $(k_x,k_y)$ is the two-dimensional wave-vector of the perturbation with wave-number $k = (k_x^2 + k_y^2)^{1/2}$, we find that the curve of marginal stability describing the in-plane compressive stress at the onset of the buckling instability $\sigma_{ajp} = -\sigma_c \delta_{ajp}$ is given by

$$\frac{\sigma_c}{E} = (kh)^2 + \frac{E_p(1-\nu_p)}{E(1+\nu)} \left( \frac{2kl+(3-4\nu_p) \sinh(2kl)}{kh[-k^2l^2+(3-4\nu_p)^2 \sinh^2(kl)]} \right)$$

(3)
If the compressive stress is larger than the value given by Eq. (3) \( A_k \) is nonzero, and the flat state exchanges stability with the periodic one. In the limit when the substrate is very deep so that \( kl >> 1 \), the most unstable wavenumber is given by minimizing Eq. (3) and yields the relation

\[
 k_c h = \left[ \frac{12(1-v_p)(1-v_p^2)}{(1+v_p)(3-4v_p)} \right]^{1/3} \left( \frac{E_p}{E} \right)^{1/3},
\]

so that the critical wavelength at onset \( \lambda_c = \frac{2\pi}{k_c} \sim h \left( \frac{E}{E_p} \right)^{1/3} \). When the applied stress exceeds that at onset of the primary buckling instability, i.e. \( \sigma = \sigma_c + \delta \sigma_c \), with \( \delta \sigma_c \sim \epsilon \sigma_c \), a band of wave-numbers around the most unstable one are excited, as shown in Figure S1(a); the width of this band scales as \( \delta k_c \sim \epsilon^{1/2} k_c \), as a simple consequence of geometry. In Fig. S1(b), we see that if we assume that the primary instability at some location has a wave-vector oriented in the \( x \)-direction, the band of excited wave-numbers in the \( x \)-direction of width \( \delta k_c \sim \epsilon^{1/2} k_c \) leads to a band of excited wave-numbers in the \( y \)-direction of width \( \delta k_y \sim \epsilon^{1/4} k_c \), which again follows from simple geometry. We thus see that above the onset of the primary instability, interactions between the excited band of wave-numbers leads to modulational instabilities on very long wavelengths of order \( O(\epsilon^{1/4}) \). The shape of the transition curve, or equivalently the linearized operator in the vicinity of the instability may then be written as

\[
 M = \epsilon \left( k - k_c \right)^2 h^2, \quad \text{with} \quad k = [(k_c + \delta k_x)^2 + \delta k_y^2]^{1/2}. \]

Expanding the previous expression and keeping only the leading order terms in \( \delta k_x \) and \( \delta k_y \) yields

\[
 M = \epsilon - \left( \delta k_x + \frac{1}{2} \frac{\delta k_x^2}{k_c} \right)^2 h^2. \]

Finally, using the substitution \( \delta k_x \rightarrow -i \partial_x \), \( \delta k_y \rightarrow -i \partial_y \), and noting that the nonlinear term follows from the phase invariance of the pattern under the transformation of the amplitude \( A \rightarrow Ae^{i\theta_0} \), then allows us obtain the equation for the complex-valued amplitude \( A(x,y) \)

\[
 \epsilon A + h^2 \left( \partial_x^2 - \frac{i}{2k_c} \partial_{yy} \right)^2 A - g|A|^2 A = 0 \tag{4}
\]
which is well-known as the Newell-Whitehead-Segel equation (S3,S4). Here $\varepsilon$ is proportional to the distance to the threshold of the instability and $g$ characterizes the saturation amplitude. Eq. (4) describes the weakly nonlinear behavior of the primary buckling pattern, and indeed provides a generic description of long wavelength modulations of cellular patterns (S1). Although a formal calculation will lead to the explicit dependence of $g$ on the problem parameters, this is of no relevance to the onset of the zigzag or the modulational instability.$^1$

A numerical simulation of Eq. (4) with periodic boundary conditions in one direction and Neumann conditions in another in a rectangular domain using a linear relaxation dynamics is shown in Fig. S1(c) and confirms the occurrence of the zigzag instability when the primary wrinkles are weakly compressed, with a series of defects or kinks corresponding to zones of double curvature. In Fig. S1(d) we show the Eckhaus instability when the wrinkles are weakly extended and in Fig. S1(e), we show the effect of a hole or other relief structure in the thin film which softens the folds in its vicinity as documented in earlier experiments (S5). In all these simulations, the strong localization of the kinks is evident, and shows how the large disparity between the energetics of bending and stretching a thin sheet favours the concentration of curvature which is energetically inexpensive. Of course Eq. (4) is no longer valid in such a region and we have to resort to a more complete description that accounts for the geometrical nonlinear behavior of the thin film (S6). Indeed, very recent work (S7, S8) shows that in the presence of external constraints, numerical simulations of Eq. (1)-(2) are able to reproduce some of these fine-scale features in complex wrinkling patterns, complementing the current work on the long wavelength features of the pattern.

References:


Figure S1

(A) The stability diagram showing the finite wavenumber buckling instability corresponding to a critical stress $\sigma_c$ for which a critical wavenumber $k_c \sim h^{-1}(E_p / E)^{1/3}$ is selected. The supercritical nature of the instability immediately characterizes the scaling behavior for the band of wavenumbers
excited when the stress is slightly larger than that required for the onset of the instability.

(B) The long wavelength modulational instability in the y-direction follows directly from the consideration of modulations of the primary instability, here assumed to be in the x-direction.

Numerical simulations (C,D,E) of (4) are done with periodic boundary conditions in the y-direction and Neumann boundaries in the x-direction. The simulations were carried out on a 128 x 128 grid with a spacing \( he^{-1/2} \), and scaling the amplitude \( A \) by \( (\epsilon/g)^{1/2} \). For all simulations we set \( k_c = 0.6 \epsilon^{1/2} / h \). We start the simulation with the amplitude \( A(x,y) = ((\epsilon - q^2)/g)^{1/2} e^{iqx} \) where a weak tensile stress perpendicular to the wrinkles corresponds to \( q < 0 \) (or equivalently a weak compressive stress along them), and evolve this initial condition using a relaxation scheme. A weak compressive stress, on the other hand corresponds to \( q > 0 \) along the wrinkles (or equivalently a weak tensile stress perpendicular to them).

(C) Numerical simulation shows the occurrence of the zigzag instability. Here the mismatch \( q = -0.1 \epsilon^{1/2} / h \) corresponds to a weak tensile stress perpendicular to the wrinkles.

(D) Numerical simulation of (4) with \( q = 0.75 \epsilon^{1/2} / h \) shows the occurrence of the Eckhaus instability, corresponding to a weak compressive stress perpendicular to the wrinkles, or equivalently, a weak tensile stress along them.

(E) Numerical simulation of (4) with \( q = -0.15 \epsilon^{1/2} / h \) shows how the zigzag instability is modified in the presence of a hole or other relief structure where we assume that the amplitude \( A \) vanishes, i.e. we have a Dirichlet boundary condition.