Dynamics of defects in the complex Ginzburg–Landau equation

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We review the derivation of basic equations for the spiral dynamics proposed by us before. These coupled equations involving the trajectories and a global phase are solved here in various situations of physical interest.

The complex Ginzburg–Landau equation (CGLE)

\[
\frac{\partial A}{\partial t} = (\mu + i\Omega)A + (1 + i\alpha)\nabla^2 A - (1 + i\beta)|A|^2A,
\]

is the normal form of an extended system making a Hopf bifurcation to an oscillating state. In (1) \(\mu\) is the bifurcation parameter and \((\alpha, \beta, \gamma)\) are real constants. This equation admits in two spatial dimensions spiral solutions (topological defects) which can disorganize the system [1]. It has been shown in ref. [2] that spiral defects appear spontaneously due to phase instability \((\gamma = 1 + \alpha\beta < 0)\) and drive the system to a disorganized state (defect-mediated turbulence). A spiral centered at the origin is of the form

\[
A(s) = D(r)\exp\{i(\theta + m\phi - S(r))\},
\]

where \((r, \phi)\) are polar coordinates and the asymptotic behavior of all the functions in (2) is known [3]. One has for \(r \to 0\) that \(D(r) = \lambda r\), \(S(r) = \mathcal{O}(r^2)\), and for \(r \to \infty\) that \(D(r) = D_\infty + \mathcal{O}(r^{-1})\), \(S(r) = qr + \gamma/2r^2 \ln r\), \(D_\infty = \sqrt{\mu - q^2}\), \(q = \beta - \alpha\), \(\gamma = q(\alpha, \beta, \mu)\) is chosen by the system, \(\omega = \Omega - \beta\mu + \nu q^2\), and \(m = \pm 1\) is the topological charge. We take the core of the spiral as \(V_c(r = 0) = \{r: |r| \leq \varepsilon, \varepsilon \approx D_\infty/\lambda\}\). The dynamics of a diluted gas of this spirals has been studied in refs. [4–10]. We want to describe for \(\gamma > 0\) a state of \(N\) spirals located at \(\{r_1(t), \ldots, r_N(t)\}\) with topological charges \(\{m_1, \ldots, m_N\}\), \(|m_j| = 1\). We look for a solution of the form

\[
A^{(s)} = (R^0 + w)\exp[i(\omega t + \Theta^0)]\]

We ask that near the \(k\)th spiral, roughly in the core \(V_c(r_k)\), \(R^0\) behaves as \(D(\rho_k)\), \(\rho_k = r - r_k\), \(\rho_k = |\rho_k|\) and \(\Theta^0\) as the phase in (2) plus corrections, i.e. the dominant term is \(\Theta^0 = m_k\varphi_k\) where \((\rho_k, \varphi_k)\) are the polar coordinates with origin at \(r_k\). Moreover the complex \(w\) is considered a small correction depending on time only through \(\{r_j(t), \varTheta^0(r, t)\}\). We assume that \(\{w, \varphi_j(t), \varTheta^0(r, t)\}\) are quantities of the same order and then \(\partial_t w = \mathcal{O}(w^2)\). We remark that in the cores \(V_c(r = r_k)\) the derivative \(\partial_t \Theta^0\) will be singular but one can also check that \(w = 0\) in the core and then terms like \(w\partial_t \Theta^0\) can be ignored. Eq. (1) has the symmetries of translation and \(A \to A \exp[i\delta]\) with constant \(\delta\); due to this single spiral solution (2) can be...
located at any point and an arbitrary constant phase can be added. When we consider a diluted gas of \( N \) spirals \( |r_i - r_j| > \epsilon \) this suggest that \( r_j(t) \) will be a slow variable as \( \Theta^0(r, t) \) outside the cores. We study this situation replacing (3) in (1). Keeping only the linear terms in \( w \) we obtain a linear equation for \( w \), of the form \( \mathcal{L}w = I \) with (\( \text{Re} \) stands for real part)

\[
\mathcal{L}w = [\mu + i(\Omega - \omega)]w + (1 + i\alpha)\nabla^2 w + 2i\nabla \Theta^0 \cdot \nabla w + [i\nabla^2 \Theta^0 - (\nabla \Theta^0)^2]w -(1 + i\beta)[2R^0 \text{ Re } w + R^0 \nabla w],
\]

where \( \mathcal{L} = \mathcal{L}^\dagger \) is the adjoint of \( \mathcal{L} \) in the same scalar product (Fredholm alternative). If \( (\alpha = \beta = 0) \) in (1) this equation is variational (real Ginzburg–Landau equation) and \( \mathcal{L} = \mathcal{L}^\dagger \), then Goldstone theorem holds in its usual way and one will obtain \( N \) approximate kernel elements \( \chi^{(k)}, k = 1, \ldots, N \), taking the derivative of \( R^0 \exp[i(\Theta^0 + \delta)] \) with respect to the position of spirals (translational symmetry) and one more element \( \chi \) by derivation with respect to \( \delta \) (symmetry \( A \rightarrow A e^{i\delta} \)). It turns out that \( \chi^{(k)} \neq 0 \) in \( V_e(r_k) \) and is almost zero outside this core while \( \chi \) is almost zero in the cores and constant in a first approximation outside the cores. We remark that the kernel elements depend on the way we parametrize the small perturbation \( w \) and what is stated above corresponds to put \( A = R^0 \exp i \Theta^0 + w \). The change to our \( w \) is straightforward (for details see ref. [6], appendix A). We assume that these properties still hold when \( \alpha \neq 0, \beta \neq 0 \). In order to determine \( \chi^{(k)} \) we consider in (4a) the dominant contribution of \( \mathcal{L}w \) in \( V_e(r_k) \) (see ref. [11])

\[
\mathcal{L}w|_{V_e(r_k)} = (1 + i\alpha)\tilde{\mathcal{L}}w = (1 + i\alpha)\left(\nabla^2 + 2i \frac{m_k}{\rho_k} \hat{\phi}_k \cdot \nabla - \frac{1}{\rho_k^2}\right)w,
\]

where \( \hat{\phi}_k = \hat{z} \times \hat{\rho}_k \), \( \hat{\rho}_k \) is the unitary vector along \( \rho_k \) and \( \hat{z} \) is the unitary vector orthogonal to the plane of the system. \( \tilde{\mathcal{L}} \) is Hermitian and \( \chi^{(k)} \) such that \( \tilde{\mathcal{L}} \chi^{(k)} = 0 \) is \( \chi^{(k)} = \exp(-im_k \Phi_k) \) as in the real case. We write \( \Theta^0 = m_k \varphi_k - S(\rho_k) + \phi^{(k)} \) near the \( k \)th spiral, where \( \phi^{(k)} \) varies slowly outside the core and \( \nabla \phi^{(k)} \) is less singular than \( L_k = \nabla[m_k \varphi_k - S(\rho_k)] \) in the core (all our assumptions can be checked a posteriori in a self-consistent way after obtaining an approximation for \( \Theta^0 \)). We evaluate now \( I \) in \( V_e(r_k) \) where \( R^0 = D(\rho_k) \). The equation for the single spiral (2) is

\[
[\mu + i(\Omega - \omega)]D(\rho_k) + (1 + i\alpha)[\nabla^2 \varphi_k D - 2iS'(\rho_k) D'(\rho_k) - iD \nabla \varphi_k S - DL_k^2] - (1 + i\beta) D^3 = 0,
\]

where

\[
\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r}
\]

and \( L_k = \nabla[m_k \varphi_k - S(\rho_k)] \).
We evaluate now $I$ from (4b) in $V_\epsilon(r_k)$ using (5). We obtain [11] for the dominant terms

$$I_{V_\epsilon(r_k)} = -j_0^* \cdot (\hat{r}_k + 2i(1 + i\alpha)\nabla\phi_k) + e^{-im_kq_k} + iD\partial_r\phi_k$$

$$+ (1 + i\alpha)D[-i\nabla^2\phi_k + (\nabla\phi_k)^2].$$

where $j_0 = (1, -im_k)$ and $u \cdot v = u_x^*v_x + u_y^*v_y$.

We impose now the solvability condition

$$(\chi^{(k)}, I) = \int V_\epsilon(r_k) \text{d}^2r = 0.$$  

Here the integral is over the whole space $\mathbb{R}^2$ but $\chi^{(k)}$ is almost zero outside $V_\epsilon(r_k)$ as we have discussed and consequently we can integrate only on the core where $\chi^{(k)} = \exp(-im_kq_k)$ and $I$ has the value (6).

The two last terms in (6) give zero since at leading order $\phi_k$ depends only on $\rho_k$ in the core as we shall see. Then (7) is given in terms of the average

$$\langle \nabla\phi_k \rangle = \frac{1}{v_\epsilon(r_k)} \int \nabla\phi_k \text{d}^2r.$$ 

Since the average on the core $\langle \nabla[m_kq_k - S(\rho_k)] \rangle_k = 0$ we have $\langle \nabla\phi_k \rangle = \langle \nabla\Theta^0 \rangle_k$ and we obtain the equation of motion of the $k$th spiral [11] ($\hat{z}$ is the unitary vector orthogonal to the plane)

$$\partial_t r_k(t) = -2m_k\hat{z} \times \langle \nabla\Theta^0 \rangle_k + 2\alpha \langle \nabla\Theta^0 \rangle_k.$$  

The next step is to obtain an equation for $\Theta^0$. For this we need $\chi \in \text{Ker } L^\dagger$ which we shall determine in the region $M = \mathbb{R}^2 - \mu \cup V_\epsilon(r_k)$ outside the cores. We can approximate outside $\mathbb{L}w$ in $M$ using $\omega = \Omega - \beta\mu + \nu q^2$, neglecting gradients in (4a) (which is correct for small $q$) and using $R^0 = D_\omega$ in $M$.

Then (4a) reduces in $M$ to $\mathbb{L}w|_M \approx -2(1 + i\beta)D_\omega^2 \text{Re } w$ at leading order. In matrix notation with $w_1 = \text{Re } w$, $w_2 = \text{Im } w$, one has

$$\mathbb{L} = -2D_\omega^2\begin{pmatrix} 1 & 0 \\ \beta & 0 \end{pmatrix}, \quad \mathbb{L}^\dagger = -2D_\omega^2\begin{pmatrix} 1 & \beta \\ 0 & 0 \end{pmatrix}$$

and $\chi = (\beta, -1)$. The solvability condition is $\beta \text{ Re } I - \text{Im } I = 0$ and from (4b) we obtain (see ref. [14])

$$\partial_t \Theta^0 = (1 + \alpha\beta)\nabla^2\Theta^0 + (\beta - \alpha)[(\nabla\Theta^0)^2 - q^2] + \mathcal{O}(q^2\nabla^2, \nabla^4).$$

This equation is valid outside the cores and in (9) we need $\Theta^0$ inside them, but we have still to impose the boundary conditions for the topological charges ($\Sigma_k = \partial V_\epsilon(r_k)$ boundary of the core)

$$\oint_{\Sigma_k} \nabla\Theta^0 \cdot dl = 2\pi m_k, \quad k = 1, \ldots, N.$$  

These conditions (12) together with (11) will determine $\nabla\Theta^0$ inside the cores. The basic equations for spiral dynamics are then (9), (11) and (12). We must remark that eqs. (9) have to be understood in an
approximate sense since the average on the core defined by (8) is not defined with precision (the size $\varepsilon$ of the core has not a precise meaning). However the coupled equations (9), (11) and (12), which we call (I) give a clear and simple picture of the relevant features of the dynamics and this is the advantage of this approach. We recall that eqs. (9) contain the variational case $\alpha = \beta = 0$ as derived by Kawasaki [12] and the nonlinear Schrödinger equation (NLSE) limit $\alpha = \beta \to \infty$ by Fetter [13]. The restriction on the validity of the coupled equations (I) comes from the phase equation (9) which is only valid for small $q$ (weak gradients outside the cores) and this means that $|\nu|/\gamma$ must be small since [3]

$$g = q \frac{\nu}{\gamma},$$

We remark that we can also derive (13) as a solvability condition for eq. (2) of the single spiral when $|\nu|/\gamma \ll 1$.

Our problem is now reduced to calculated $\langle \nabla \Theta^0 \rangle_k$ in (9). The equation was first written by us in ref. [4] but with $\nabla \phi^{(k)}$ instead of $\langle \nabla \Theta^0 \rangle_k$ which is not important since at leading order $\langle \nabla \Theta^0 \rangle_k = \nabla \phi^{(k)}$ in ref. [4] we have remarked that $\phi^{(k)}$ was the total phase $\Theta^0$ minus the self-phase of the $k$th spiral. However, as we have discussed in ref. [5], our calculation of $\langle \nabla \Theta^0 \rangle_k$ was based in the assumption that in order to satisfy the boundary condition (12) $\phi^{(k)}$ had the form

$$\phi^{(k)} = \sum_{i \neq k} [m_i \varphi_i - \tilde{S}(\rho_i)] + \Psi(r, t), \quad \tilde{S}(\rho_i) = S(\rho_i) - q\rho_i,$$

and that $\Psi$ could be neglected. This assumption turns out to be unjustified as we shall see by solving explicitly the phase equation (11) with (12). We shall solve (I) first for $\nu = 0$ ($\alpha = \beta$) and then $\nu$ small.

When $\nu = 0$ (which corresponds to vortices since $q = 0$) eq. (11) reduces to

$$\partial_t \Theta^0 = (1 + \alpha^2) \nabla^2 \Theta^0. \quad (14)$$

In the NLSE limit ($\alpha \to \infty$) this reduces to $\nabla^2 \Theta^0 = 0$ which has the solution $\Theta^0 = \Sigma_{i=1}^N m_i \varphi_i$ satisfying (12). This gives the usual dynamics of hydrodynamical vortices and eq. (1) is reduced to the dynamics of $N$ points.

For finite $\alpha$ the solution $\tilde{\Theta}^0$ of (14) satisfying (12) is given by [14] ($\nu_j = \dot{r}_j/(1 + \alpha^2)$)

$$\nabla \tilde{\Theta}^0 = -\hat{z} \times \sum_{j=1}^N m_j \exp [-\frac{1}{2} \nu_j \cdot (r - r_j)](\frac{1}{2} \nu_j + \nabla) K_0(\frac{1}{2} \nu_j |r - r_j|), \quad (15)$$

where $K_0(x)$ is the usual Bessel function [15]. We can calculate explicitly the average $\langle \nabla \Theta^0 \rangle_k$ from (15) under the condition $r_{ij} = |r_{ij}| = |r_j - r_i| >> \varepsilon$ (diluted gas of vortices) and replacing in (9) we obtain the coupled equations for the $N$ vortices. We consider the case of two vortices ($N = 2$) and after scaling space as $Q = \varepsilon^{-1} r$ and time as $\tau = 4(1 + \alpha^2)\varepsilon^{-2} t$ the size $\varepsilon$ of the core disappears from the equations. Let $Q = \frac{1}{2}(Q_1 + Q_2)$ be the center of mass coordinate and $u = Q_1 - Q_2$ the relative coordinate. In the case $\alpha = 0$ we obtain after some long calculations (see ref. [14] for details) that the center of mass does not move and that the dominant term in the equation of motion is

$$|\dot{u}| \ln |\dot{u}| = -\frac{1}{u}. \quad (16)$$
For equal changes $m_1 = m_2$ one has repulsion ($\dot{u} > 0$) and for $m_1 \neq m_2$ attraction ($\dot{u} < 0$) and the range of validity is $u \gg 1$. When $\alpha = 0$ one obtains [14]: (a) for $m_1 = m_2$ one has repulsion, the center of mass does not move but the vortices rotate; (b) for $m_1 \neq m_2$ one has attraction, there is no rotation but now the center of mass moves perpendicular to the relative velocity ($\dot{Q} \cdot \dot{u} = 0$). In the NLSE limit $\alpha \to \infty$ we put $\dot{X} = u\dot{\phi}$, $n = 1$ for $m_1 = m_2 = m$ ($\dot{u} = \dot{u} + u\dot{\phi}$) and $\dot{X} = 2\dot{Q}$, $n = -1$ for $m_1 = -m_2 = m$. After scaling time we obtain up to first order $\mathcal{O}(\alpha^{-1})$

$$\dot{X} = \frac{mn}{\alpha} \frac{1}{u}, \quad \dot{u} = \frac{n}{2} \frac{1}{\alpha} \frac{u}{u}.$$  

We must point out here that in order to obtain vortices in this limit ($\alpha \to \infty$) we must take $\Omega = \sigma\alpha$ ($\sigma$ is a constant) in (1). The frequency of our vortices tends to infinity and consequently the size of the core goes to zero. It is to these vortices that our results apply. In fact Y. Pomeau has shown in this conference that in the NLSE the phase equation is a wave type equation and this modify dramatically the evolution of vortices. Our calculation correspond to the case when the speed in the wave type equation goes to infinity since $\Omega \to \infty$ in our limit. The results of Pomeau are related to the finite size of the core and disappear when the vortices become pointlike which is the case we attain in the limiting procedure.

The case of one vortex in an external phase gradient $k$ can be easily treated in the same way [14] and one reproduces the known result [16-18]

$$\ln(\frac{1}{e} |\hat{r}_i|) \hat{r}_i(t) = 2m_i \hat{z} \times k.$$  

The factor $\frac{1}{e}$ in this last equation is not to be taken as exact in agreement with our discussion about the basic equation (9) in the paragraph after eq. (12). The important fact here is the appearance of the size of the core $e$ in eq. (18). Using matching methods the numerical factor can be calculated [16, 17] and comparison with (18) allows an estimation of the core. The same comments apply to (16) where $e$ is not present due to our scaling (if the factor is different we can modify the scaling to obtain (16) but the important fact is the dependence on $e$ of the scaling).

We turn now to the case $\nu = \beta - \alpha \neq 0$ but small. Putting $\Theta^0 = (\gamma/\nu) \ln G$ the phase equation (11) becomes linear:

$$\frac{1}{\gamma} \partial_t G = (\nabla^2 - g^2) G, \quad g = q \frac{\nu}{\gamma}. \tag{19}$$

The boundary condition (12) is

$$\oint_{\Sigma_k} d(\ln G) = \frac{\nu}{\gamma} 2\pi m_k \tag{20}$$

and the equation of motion (9) becomes

$$\frac{\nu}{\gamma} \partial_t r_k(t) = -2m_k \hat{z} \times (\nabla(\ln G))_k + 2\alpha (\nabla(\ln G))_k.$$  

The essential observation proposed by Pismen and Nepomnyashchy [10] is to notice that the boundary condition (20) comes in higher order in $\nu/\gamma$ which we are assuming small. We forget then (20) in a first
approximation and look for a suitable solution $G^0$ of (19). Then (21) tells us that in lowest order the equation of motion is simply

$$\langle \nabla (\ln G^0) \rangle_k = 0. \quad (22)$$

The solution $G^0$ of (19) is

$$G^0 = \sum_i \exp \left(-\frac{\dot{r}_i}{2\gamma} \cdot (r - r_i)\right) K_0 \left(g_i |r - r_i|\right), \quad (23)$$

with $g_i^2 = g^2 + r_i^2/4\gamma^2 \approx g^2$. Replacing in (22) after the average on the core we obtain $K_1(x) = -K_0(x)$

$$\frac{\dot{r}_k}{2\gamma} = \mu_k \sum_{i \neq k} \left(-\frac{\dot{r}_i}{2\gamma} K_0 (g_i r_{ik}) - g_i K_1 (g_i r_{ik}) \dot{r}_{ik}\right). \quad (24)$$

The constant $\mu_k$ turns out be independent of $k$ and has the value

$$\mu_k = \left.<\exp\left[-\left(\frac{\dot{r}_k}{2\gamma}\right) \cdot (r - r_k)\right]\right>_k = \frac{2}{\pi} \frac{|\nu|}{\gamma} + o\left(\frac{|\nu|^2}{\gamma}\right), \quad (25)$$

where we have used (13). Keeping the dominant term in (24) ($\dot{r}_i$ is exponentially small) we obtain at leading order

$$\dot{r}_k = -\frac{4}{\pi} \nu g \sum_{i \neq k} K_1 (g r_{ik}) \dot{r}_{ik}, \quad (26)$$

where $K_1(x) = e^{-x}/\sqrt{x}$, $x \gg 1$. For two spirals we conclude that the center of mass does not move and if $\dot{u} = u \ddot{u}$ is the relative velocity one has

$$\ddot{u} = -\frac{8}{\pi} \nu g K_1 \left(g u\right), \quad (27)$$

which is exponentially small, always attractive, and agrees with ref. [10] and with the previous result of Aranson, Kramer and Weber [8,9]. It is simple to see that the motion of the center of mass and the rotation comes at the next order of the approximation. This is consistent with the numerical simulations in [8,9]. Although $\nabla (\ln G^0)$ is singular near the position $r_k$ of the $k$th spiral as $1/\rho_k \ln \rho_k$ the most singular behavior of $\nabla \Theta^0$ in the core $V_\varepsilon (r_k)$ will come from the next term $G^1$ if we put $G = G^0 + (\nu/\gamma) G^1$. One has in $G \approx \ln G^0 + (\nu/\gamma) G^1 / G^0$ and $(\nu/\gamma) (G^1 / G^0) \approx m_k q_k$ in $V_\varepsilon (r_k)$ in order to satisfy (20) (notice that this behavior was used to derive (9)) and then $\nabla G^1 / G^0 = o(\rho_k^{-1})$ which dominates $\nabla \Theta^0$. We remark that this term does not contribute to the equation of motion (9) of the spiral since $\langle \nabla (m_k q_k) \rangle_k = 0$. We conclude then that the basic equations for spiral dynamics are the system of coupled equations (9), (11) and (12) proposed in refs. [4] and [5].

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