Dynamics of spiral rings in the three dimensional Ginzburg–Landau equation

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In this paper, we show that the spiral rings of the 3D Ginzburg–Landau equation shrink out and disappear in finite time. In contrast with the 2D case in which the interaction between two spirals of opposite charge is exponential. We show analytically and numerically that the spiral rings collapse with a power law due to the continuity of the defect line.

The problem of the interaction of spiral defects in two dimensional systems has been of considerable interest recently [1–8]. It is known that spiral defects of the complex Ginzburg–Landau equation (CGLE) (see eq. (1)) in two dimensional systems interact via an exponentially small “force” [5–8]. In 3D this picture is completely different. First of all, defects are not points but closed lines. In order to study the three dimensional world of the CGLE we did the following numerical experiment. We simulated the CGLE on the massively parallel computer (The Connection Machine) using finite difference algorithm and a grid of 128 x 128 x 128 pixels. As in the 2d case we observed defect-mediated turbulence [9] for 1 + \alpha \beta < 0. Furthermore we expected to freeze the defect as in 2d [13] by rapidly quenching the systems into phase stability (1 + \alpha \beta > 0). However, we were very surprised to observe that all the defect disappeared from the box in a finite time.

In order to understand this phenomenon we did the following experiment. We used as initial conditions a spiral ring which is the generic defect of the CGLE and we observed that the radius of the ring shrunk out in a finite time to zero (see fig. 1) leaving a target wave in the system.

We consider the CGLE as

$$\frac{\partial A}{\partial t} = \mu A + (1 + i \alpha) \nabla^2 A - (1 + i \beta)|A|^2 A,$$

(1)

where A is a complex order parameter, \mu is the bifurcation parameter of a Hopf bifurcation and (\alpha, \beta) are real constants. The defect position is well described by two parameters the time t and a line parameterization \sigma i.e. r(\sigma, t). As we turn around the defect line the phase of A changes by \pm 2\pi. Roughly speaking, based on the two dimensional case, a point of the ring does not interact with the diametrically opposite points but we can expect some non-negligible interaction with the neighbouring points.

Now we present a naive theory of the self interaction of a spiral loop based on the results of the interaction of spiral defects in two dimensions, see ref. [8] for details. In two dimensions the motion of the defects is coupled to the phase equation which is valid far from the core. We have in mind that for spiral rings whose radius of curvature is big \#1 the dynamic of a defect in the 2D case can be locally applied in 3D, i.e. The defect line can be locally approximated by a

\#1 In fact the radius R must be bigger than the core radius which is of the order 1/\sqrt{\mu}.
straight line since the radius of curvature is big. Therefore the modulus of the defect-speed of the segment is approximately [1–3,8]

\[ |\partial_r(\sigma, t)| \approx |\nabla \Theta| ; \]  

(2a)

furthermore the speed direction is always orthogonal to the defect line. The phase-equation is the well known Kuramoto equation in 3D:

\[
\partial_t \Theta = (1 + \alpha \beta) \nabla^2 \Theta + (\beta - \alpha) |(\nabla \Theta)^2 - q^2| + \mathcal{O}(q^2 \nabla^2, \nabla^4),
\]

(2b)

where \( q \) is the wave vector far away from the defect line. In 2D Hagan [10] developed a relation between \( q \) and \( (\alpha, \beta) \); however, in 3D we do not know the explicit value of \( q \). The validity of eq. (2) is only for \( q^2 \ll \mu \), i.e. when we are
near the variational case. Now we introduce the “non-variational parameter” \( s = (\beta - \alpha)/(1 + \alpha \beta) \) so that if \( s < 1 \) then \( q^2 \ll \mu \) [8,10]. This is also verified in 3D.

Via the Cole–Hopf transformation \( \Theta = (1/s) \times \ln G \) the phase equation (2b) becomes linear,

\[
\frac{1}{1 + \alpha \beta} \partial_t G = (\nabla^2 - g^2) G, \quad g = qs, \tag{3}
\]

but the boundary condition in the line defect becomes more complicated (nonlinear); however using the Pismen and Nepomnyashchy linear expansion \( G = G^0 + s G^1 + s^2 G + \cdots \) one notes that the topological boundary condition comes in at higher order in \( s \), which we are assuming small, i.e. a spiral can be considered as a target at first order. The equation of motion for the line (2a) at first order will be [8]

\[
\nabla \ln G^0 = 0, \tag{4}
\]

where \( \nabla \ln G^0 \) is evaluated on the line defect. We notice here that in 3D there is no singularity when we carry out this operation unlike the 2D case. Eq. (4) is the condition that will give us \( \dot{r}(\sigma, t) \) for \( G^0 \) a regular solution of eq. (3).

The solution of (3) without branches (the target solution in 3D) is

\[
G^0(x, t) = \int d\sigma \exp \left( -\frac{\dot{r}(\sigma', t) \cdot (x - r(\sigma', t))}{2(1 + \alpha \beta)} \right) \times Y(g_{\sigma'}, |x - r(\sigma', t)|), \tag{5}
\]

where \( Y(z) \) is the Yukawa function \( Y(z) = e^{-z}/z \) and \( g_{\sigma'}^2 = g^2 + \dot{r}(\sigma', t)^2/4(1 + \alpha \beta)^2 = g^2 \). We have neglected terms in \( \partial_{\sigma} r(\sigma', t) \). If we calculate the gradient of \( G^0 \) and evaluate it at \( r = r(\sigma, t) \), (4) transforms into

\[
\nabla \ln G^0|_{\sigma} = \int d\sigma' \exp \left( -\frac{\dot{r}(\sigma', t) \cdot \rho_{\sigma\sigma'}}{2(1 + \alpha \beta)} \right) \times \left( -\frac{\dot{r}(\sigma', t) }{2(1 + \alpha \beta)} Y(g \rho_{\sigma\sigma'}) + g Y'(g \rho_{\sigma\sigma'}) \dot{\rho}_{\sigma\sigma'} \right)
\]

\[
= 0, \tag{6}
\]

where \( \rho_{\sigma\sigma'} = r(\sigma, t) - r(\sigma', t) \). It is difficult to solve for the line motion \( \dot{r}(\sigma, t) \) from (6) because it is an integral equation. To study the motion of a spiral ring of radius \( R(t) \) we used a particular point \( r(\sigma = 0, t) = (R(t), 0, z(t)) \). A spiral ring can be parameterized by \( r(\sigma', t) = (R(t) \cos \sigma', R(t) \sin \sigma', z(t)) \). With the above definitions \( \rho_{\sigma\sigma'} = 2R(t) \sin \frac{1}{2} \sigma' \) \( (\sin \frac{1}{2} \sigma', -\cos \frac{1}{2} \sigma', 0) \). This introduces some simplifications in (6): the \( j \) component does not give any information, but the \( k \) projection gives \( z(t) = 0 \), i.e. there is no drift at order zero (this is consistent with numerical measurements). Finally the projection on the \( i \) component gives the integral relation

\[
\int d\sigma' \exp \left( \frac{R \dot{R} \sin^2 \frac{1}{2} \sigma'}{1 + \alpha \beta} \right) \left( -\frac{\dot{R} \cos \sigma'}{2(1 + \alpha \beta)} Y(2gR \sin \frac{1}{2} \sigma') + g Y'(2gR \sin \frac{1}{2} \sigma') \sin \frac{1}{2} \sigma' \right) = 0. \tag{7}
\]

Using the identities \( \cos \sigma' = 1 - 2 \sin^2 \frac{1}{2} \sigma' \) and \( Y'(z) = -Y(z) (1 + 1/z) \), we can compute the dominant term of the integral in (7) which leads to the equation

\[
\dot{\partial}_i R(t) = -\frac{1 + \alpha \beta}{R}, \tag{8}
\]

\[\text{Fig. 2. Diameter of the spiral versus the time (} \mu = 1, \alpha = 0 \text{ and } \beta = 1 \text{).}\]
In (t* - t) Fig. 3. Ln of the diameter of the spiral versus the ln(t* - t); we found a potential law \( R(t) \approx (t^* - t)^\nu \), \( \nu = 0.48 \).

whose solution is

\[
R(t) = \sqrt{2(1 + \alpha \beta)}(t^* - t)^{1/2}.
\]

Therefore \( R = 0 \) at \( t = t^* \), where \( t^* \) is the collapse time related to the initial conditions by \( t^* = R^2(0)/2(1 + \alpha \beta) \). This law has been confirmed by our numerical simulation (see figs. 2 and 3).

Finally we can see that at first order the dynamics of the ring (8) does not depend on the wave vector \( q \). We therefore expect the annihilation law of a vortex ring in the variational case (\( \alpha = \beta \)) to be the same. We have verified this property both analytically and numerically. In the conservative case, when the CGLE reduces to the nonlinear Schrödinger equation (defocusing), Y. Pomeau [11,12] has shown that the vortex ring still collapses but at a different rate.

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References