Analytical description of a state dominated by spiral defects in two-dimensional extended systems

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We study the state of a diluted gas of spiral defects observed in a recent simulation of a two-dimensional extended system governed by the Ginzburg-Landau equation with complex coefficients. We show that this state, which is responsible for the disorganization of the system, can be described by an approximate solution in which the new variables are the positions of the spirals and a new global phase. The dynamics of these new variables and the conditions for the existence of the state are determined.

1. Introduction

The role of defects in the disorganization of two-dimensional extended systems has recently been considered by several authors [1, 2]. In the case of spiral defects which appear in the Ginzburg-Landau equation with complex coefficients Coullet, Gil and Lega [2] have shown that when the phase is unstable, a state dominated by defects establishes which is characterized by a constant density of spirals and a rapid decrease of the correlation function. It is this state which we want to describe here.

We do this assuming that there exists a solution of the Ginzburg-Landau equation which has a dominating term describing the state dominated by well-separated spirals plus a small correction (the distance between the spirals is much larger than the core of the defects, i.e. we have a diluted gas of spirals). In the construction of this ansatz we rely on the similar problem in the real Ginzburg-Landau equation where Goldstone's theorem indicates the existence of three types of marginal modes associated with translational invariance along the two axes and with phase invariance. We expect then that the new relevant variables will be the positions of the centers of the spirals \( (r_1(t), r_2(t), \ldots, r_N(t)) \), associated with the translational invariance, and a new global phase \( \Psi(r, t) \), associated with the phase invariance. The equations of motion for these new variables are then obtained through a solvability condition for the small correction to the dominating term of the ansatz. This program is carried out in section 2, where the equations for \( r_k \) and \( \Psi(r, t) \) are determined. A short account of these results was given by us in ref. [3]. These equations of motion have also been obtained independently by Elphick and Meron [4].

In section 3 we study some consequences of the equations of motion and compare with the numerical experiment in ref. [2]. Appendix A treats the real Ginzburg-Landau equation and appendix B gives a discussion of the stability of a single spiral in the external region. In appendix C we extend our results to a more general class of equations admitting spiral solutions (the \( \Lambda-\Omega \) systems) and finally in appendix D we discuss the equations of motion for spirals of arbitrary charge.
We can summarize saying that our results provide an analytical description of the numerical experiment in ref. [2]. Our equations of motion predict also the observed behavior of the motion of spirals observed in a recent simulation [5].

2. From Ginzburg–Landau equation to spiral dynamics

We write the complex Ginzburg-Landau equation in the form

\[ \partial_t A = \mu A + (1 + i\alpha) \nabla^2 A - (1 + i\beta)|A|^2 A, \] (2.1)

which after the transformation \( A = A \exp(-i\alpha t) \) becomes

\[ \partial_t A = (1 + i\alpha) \left( \mu A + \nabla^2 A - \frac{\gamma + i\nu}{1 + \alpha^2} |A|^2 A \right) \] (2.2)

with \( \gamma = 1 + \alpha \beta, \nu = \beta - \alpha \). The spiral solution of (2.2) is [6–8]

\[ A(r, \varphi, t) = D(r) \exp \{i[\omega t + m\varphi - S(r)]\}, \] (2.3)

where \( m \in \mathbb{Z} \), and \((r, \varphi)\) are the polar coordinates in the plane. Introducing (2.3) in (2.2) we obtain for \( D(r) \) and \( S(r) = \frac{dS}{dr} \) coupled differential equations

\[ \left( \mu - \frac{\alpha \omega}{1 + \alpha^2} \right) D(r) + \nabla_r^2 D - \left( \frac{m^2}{r^2} + S_r^2 \right) D - \frac{\gamma}{1 + \alpha^2} D^3 = 0, \] (2.4a)

\[ 2S_r D_r + \nabla_r^2 SD + \frac{\omega}{1 + \alpha^2} D + \frac{\nu}{1 + \alpha^2} D^3 = 0, \] (2.4b)

where

\[ D_r = \frac{dD}{dr}, \quad \nabla_r^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r}. \]

We need three boundary conditions to solve (2.4) and we take

\[ D(0) = 0, \quad S_r(0) = 0 \quad \text{and} \quad S_r(r \to \infty) = q. \] (2.5)

The system chooses \( q \), which is then a function of \( \alpha \) and \( \beta \) (in section 3 we study the dependence on \( q \)). Imposing the asymptotic condition in the external region we find

\[ D(r \to \infty) = D_\alpha = \sqrt{\mu - q^2} \quad \text{and} \quad \omega = -\nu D_\alpha^2 \] (2.6)

and we can write eqs. (2.4) in terms of \( D_\alpha \) and \( q \),

\[ \nabla_r^2 D - \left( \frac{m^2}{r^2} + S_r^2 - q^2 \right) D + \frac{\gamma}{1 + \alpha^2} \left( D_\alpha^2 - D^2 \right) D = 0, \] (2.7a)

\[ 2D_r S_r + D \nabla_r^2 S = \frac{\nu}{1 + \alpha^2} \left( D_\alpha^2 - D^2 \right) D. \] (2.7b)
We consider from now on one-armed spirals \(|m| = 1\) and remarking that

\[ 2D_s S_r + D \nabla^2 S = \frac{1}{rD} \frac{d}{dr} (rD^2 S_r) \]

eqs. \((2.7)\) give the asymptotic behavior

\[ D(r) \sim \lambda r, \quad r \to 0; \quad D(r) \sim \frac{q(1 + \alpha^2)}{2\nu D_\infty} \frac{1}{r}, \quad r \to \infty, \]

\[ S_r(r) \sim \frac{\nu D_\infty^2}{1 + \alpha^2} \frac{1}{2\nu} r, \quad r \to 0; \quad S_r \to q + \frac{\gamma}{2\nu} \frac{1}{r}, \quad r \to \infty. \]

The function \(D(r)\) attains its asymptotic value after a distance of order \(\epsilon\). We call the circle \(V_r(r = 0)\) of radius \(\epsilon\) around the origin the core of the spiral.

Here \(\lambda\) and \(q\) are functions of \((\alpha, \beta)\) and \(q\) vanishes for \(\nu = 0\), which corresponds to vortex type solutions where one also has \(S_r = 0\) \([6]\). The amplitude of the spiral solution \((2.3)\) vanishes at \(r = 0\) and the circulation of its phase around the origin is \(m\), reciprocally a solution of \((2.2)\) which vanishes at the origin behaves locally as a spiral.

Our purpose now is to describe a solution of \((2.2)\) representing a diluted gas of \(N\) spirals located at points \(r_1, r_2, \ldots, r_N, \quad |r_i - r_j| \gg \epsilon\), in other words the solution we are looking for vanishes in these points. We write the ansatz

\[ A(r, t) = (R^0 + w) \exp[i(\omega t + \Theta^0)], \]

where

\[ R^0 = \sum_{i=1}^{N} [D(\rho_i) - D_\infty] + D_\infty, \quad \rho_i = r - r_i, \quad \rho_i = |\rho_i|, \]

\[ \Theta^0 = \sum_{i=1}^{N} [m_i \varphi_i - \bar{S}(\rho_i)] - q \min(\rho_1, \rho_2, \ldots, \rho_N) + \Psi(r, t). \]

Here \(m_i \in \mathbb{Z}, \quad \bar{S}(\rho_i) = S(\rho_i) - q \rho_i, \quad \cos \varphi_i = (x - x_i)/\rho_i, \quad \sin \varphi_i = (y - y_i)/\rho_i, \quad r = (x, y), \quad r_i = (x_i, y_i)\) and \(\Psi(r, t)\) is a global new phase we have to determine. In order to show how \((2.9)\) represents the actual situation we have plotted in fig. 1 the ansatz for \(N = 4\). We allow the positions of the spirals to move, thus \(r_i\) become functions \(r_i(t)\), and \(w \in \mathbb{C}\) is a small correction term whose time dependence occurs only via \(\{r_i(t), \Psi(r, t)\}\), which are the new relevant variables for the situation we want to study. The motivation for the ansatz is that in a region around each point \(r_i\), center of the core of the \(i\)th spiral, we want the dominant terms of the solution \((2.9)\) to behave like a spiral centered in \(r_i\). The region around the \(i\)th spiral is in fact defined by the minimum \(\min(\rho_1, \rho_2, \ldots, \rho_N)\) of the set \(\{\rho_1, \rho_2, \ldots, \rho_N\}\), and the inclusion of this term in \(\Theta^0\) guarantees the behavior we want. We call \(D_k\) the region defined by \(D_k = \{r: \min(\rho_1, \rho_2, \ldots, \rho_N) = \rho_k\}\).
Introducing the ansatz (2.9) in (2.2), and using (2.6), we obtain

\[
\partial_t R^0 + i R^0 \partial_t \Theta^0 = (1 + i \alpha) \left( \frac{\gamma + i \nu}{1 + \alpha^2} \left( D^2 - |R^0 + w|^2 \right)(R^0 + w) + \nabla^2 (R^0 + w) \right)
- \left( R^0 + w \right) \left( (\nabla \Theta^0)^2 - q^2 \right) + 2i \nabla \left( R^0 + w \right) \cdot \nabla \Theta^0 + i \left( R^0 + w \right) \nabla^2 \Theta^0 ,
\]

(2.11)

where we have neglected

\[
\partial_t w = \sum_i \frac{\partial w}{\partial r_i} + \int d r' \dot{\Psi} (r', t) \frac{\partial w}{\partial \Psi}
\]

because we assume that \((r_i, \Psi(r, t))\) are small quantities of the same order as the correction \(w\) and consequently \(\partial_t w = \mathcal{O}(w^2)\). Keeping only linear terms in \(w\) in (2.11) this equation takes the form

\[
\mathcal{L} \left( \begin{array}{c} \text{Re } w \\ \text{Im } w \end{array} \right) = I ,
\]

with

\[
\mathcal{L} = \begin{pmatrix}
g - \frac{2 \gamma (R^0)^2}{1 + \alpha^2} + \nabla^2 - \left[ (\nabla \Theta^0)^2 - q^2 \right] & -n - 2 \nabla \Theta^0 \cdot \nabla - \nabla^2 \Theta^0 \\n- \frac{2 \nu (R^0)^2}{1 + \alpha^2} + 2 \nabla \Theta^0 \cdot \nabla + \nabla^2 \Theta^0 & g + \nabla^2 - \left[ (\nabla \Theta^0)^2 - q^2 \right] \end{pmatrix}
\]

(2.12)
and

\[
I = \left( \frac{1}{1 + \alpha^2} \left( \partial_r R^0 + \alpha R^0 \partial_r \Theta^0 \right) - g R^0 - \nabla^2 R^0 + R^0 \left( (\nabla \Theta^0)^2 - q^2 \right) \right)
\]

\[
\left( \frac{1}{1 + \alpha^2} \left( -\alpha \partial_r R^0 + R^0 \partial_r \Theta^0 \right) - n R^0 - 2 \nabla R^0 \cdot \nabla \Theta^0 - R^0 \nabla^2 \Theta^0 \right),
\]

(2.13)

where

\[
g = \frac{\gamma}{1 + \alpha^2} \left( D^2 - (R^0)^2 \right) \quad \text{and} \quad n = \frac{\nu}{1 + \alpha^2} \left( D^2 - (R^0)^2 \right).
\]

In the space of functions \(\{g_\mu(r), \mu = 1, 2\}\) where \(\mathcal{L}\) acts, we define the scalar product

\[
\langle \chi^{(1)}, \chi^{(2)} \rangle = \int dr \sum_{\mu=1}^{2} g_\mu \chi^{(1)}_\mu \chi^{(2)}_\mu.
\]

The solvability condition for \(\mathcal{L} w = I\) is then that \(I\) is orthogonal to the kernel of the adjoint \(\mathcal{L}^*\) of \(\mathcal{L}\), i.e. \(\langle \chi, I \rangle = 0\) for \(\chi \in \text{Ker } \mathcal{L}^*\). The solvability conditions will determine the equations of motion of the new variables \(\{r_i(t), \Psi(r, t)\}\), i.e. will give the values of \(\dot{r}_i(t)\) and \(\Psi(r, t)\). Our problem now is to determine \(\text{Ker } \mathcal{L}^*\), where

\[
\mathcal{L}^* = \left( \begin{array}{c}
g - \frac{2 \gamma (R^0)^2}{1 + \alpha^2} + \nabla^2 - \left[ (\nabla \Theta^0)^2 - q^2 \right] \\
- n + 2 \nabla \Theta^0 \cdot \nabla + \nabla^2 \Theta^0 \\
g + \nabla^2 - \left[ (\nabla \Theta^0)^2 - q^2 \right]
\end{array} \right).
\]

(2.14)

Our guide to determine a basis of the space \(\text{Ker } \mathcal{L}^*\) will be the analysis we give in appendix A of the motion of the vortex solutions of the Ginzburg–Landau equation with real coefficients. The results presented there suggest that we should find \(2N\) vectors \(\{X_{i}^{(k)}(r), k = 1, \ldots, N\}\) of \(\text{Ker } \mathcal{L}^*\) associated to the invariance of the original equation (2.2) under translations and another vector \(X_{III}\) associated to the invariance of (2.2) under \(A \rightarrow A e^{i \delta}\), where \(\delta\) is a constant phase. Let \(V_{i}(r, t)\) be the circle of radius \(\epsilon\) centered at \(r_i\) (core of the \(k\)th spiral). The vectors \(\{X_{i}^{(k)}, X_{II}^{(k)}\}\) should vanish outside this core \(V_{i}(r, t)\) (see (A.15) and (A.16)) and determine the motion of the \(k\)th spiral. Concerning \(X_{III}\) this vector should vanish in the cores \(V_{j}(r, t), j = 1, \ldots, N\) (see (A.19) and notice that \(\phi_n^j\) is proportional to \(D(|r - r_j|)\) in the neighborhood of \(V_{j}(r, t)\). The condition \(\langle X_{III}, I \rangle = 0\) will determine the evolution equation for the new global phase \(\Psi(r, t)\), which we assume to be a slowly varying function as well as its time derivative \(\partial_t \Psi(r, t)\), in particular we can consider these functions as approximately constant through distances of order \(\epsilon\) (the dimension of the core of a spiral). Our method to determine \(\chi \in \text{Ker } \mathcal{L}^*\) will be simply to approximate \(\mathcal{L}^*\) in the region where \(\chi \neq 0\) and then to solve. As we shall see this strategy works here in both cases of interest: (a) \(\{X_{I}^{(k)}, X_{II}^{(k)}\} \neq 0\) only on \(V_{k}(r, t)\); (b) \(X_{III} \neq 0\) only outside the cores \(V_{j}(r, t), j = 1, \ldots, N\).
In the region $D_k$ around the $k$th spiral it is convenient for what follows to write

$$R^0 = D(\rho_k) + \sum_{i \neq k} \left[ D(\rho_i) - D_\omega \right],$$

(2.15a)

$$\Theta^0 = m_k \phi_k - S_r(\rho_k) + \Phi^{(k)}(r, t),$$

(2.15b)

$$\Phi^{(k)}(r, t) = \sum_{i \neq k} \left[ m_i \phi_i - \bar{S}(\rho_i) \right] + \Psi(r, t).$$

(2.15c)

We put

$$L_k = \frac{m_k}{\rho_k} \dot{\phi}_k - S_r(\rho_k) \hat{\rho}_k,$$

(2.16)

where $\hat{\rho}_k$ is the unitary vector along $\rho_k$ and $\dot{\phi}_k = \dot{z} \wedge \hat{\rho}_k$ where $\dot{z}$ is a unitary vector orthogonal to the plane of the system. We have (the prime denotes derivative with respect to the argument)

$$\nabla \Theta^0 = L_k + \nabla \Phi^{(k)}(r, t),$$

(2.17a)

$$\nabla^2 \Theta^0 = -\nabla_{\rho_k}^2 S(\rho_k) + \nabla^2 \Phi^{(k)}(r, t),$$

(2.17b)

$$\partial_t \Theta^0 = -L_k \cdot \dot{r}_k + \partial_t \Phi^{(k)}(r, t),$$

(2.17c)

$$\partial_t R^0 = -\sum_i D'(\rho_i) \hat{\rho}_i \cdot \dot{r}_i, \quad \nabla R^0 = \sum_i D'(\rho_i) \hat{\rho}_i,$$

(2.17d)

$$\nabla^2 R^0 = \sum_i \nabla^2 D(\rho_i).$$

(2.17e)

In order to determine $\{\chi_1^{(k)}, \chi_\Pi^{(k)}\}$ in the core $V_\epsilon(r_k)$ we keep in $\mathcal{E}^+ \setminus \text{core}$ the most singular elements inside the core; this gives

$$\mathcal{E}^+ \approx \begin{pmatrix} \nabla^2 - \frac{m_k^2}{\rho_k^2} - 2\left( \frac{m_k}{\rho_k} \right) \hat{\phi}_k \cdot \nabla & -2\left( \frac{m_k}{\rho_k} \right) \hat{\phi}_k \cdot \nabla^2 \frac{m_k}{\rho_k^2} \\ 2\left( \frac{m_k}{\rho_k} \right) \hat{\phi}_k \cdot \nabla & \nabla^2 - \frac{m_k^2}{\rho_k^2} \end{pmatrix},$$

(2.18)

and the solutions of $\mathcal{E}^+ \chi = 0$ are

$$\chi_1^{(k)}(\rho_k) = \begin{pmatrix} \cos m_k \phi_k \\ -\sin m_k \phi_k \end{pmatrix} + \mathcal{O}(\rho_k^2),$$

(2.19a)

$$\chi_\Pi^{(k)}(\rho_k) = \begin{pmatrix} \sin m_k \phi_k \\ \cos m_k \phi_k \end{pmatrix} + \mathcal{O}(\rho_k^2).$$

(2.19b)

These vectors correspond, respectively, to $\partial \Xi^{(k)}/\partial x_k$ and $\partial \Xi^{(k)}/\partial y_k$ in the case of the real Ginzburg–Landau equation as discussed in appendix A. The conditions $\langle \chi_1^{(k)}, I \rangle = 0$ and $\langle \chi_\Pi^{(k)}, I \rangle = 0$ give $\dot{x}_k$ and $\dot{y}_k$. Explicitly these conditions take the form

$$\int_{V_\epsilon(r_k)} d\rho_k \begin{pmatrix} l_1 \cos m_k \phi_k \\ l_2 \sin m_k \phi_k \end{pmatrix} + l_2 \begin{pmatrix} -\sin m_k \phi_k \\ \cos m_k \phi_k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

(2.20)
where the first line corresponds to \( \chi^{(k)}_1 \) and the second \( \chi^{(k)}_2 \), \( d\rho_k = \rho_k d\rho_k d\varphi_k \), and the region of integration is the core \( V'_c(r_k) \) since \( \{\chi^{(k)}_1, \chi^{(k)}_2\} \) vanish outside this region. We replace in (2.20) the expression of \( I \) given by (2.13) and use formulas (2.17), then we remark that all integrals not containing \( \hat{\rho}_k \) or \( \hat{\varphi}_k \) vanish since the integrand is a slowly varying function of \((\rho_k, \varphi_k)\) according to our assumptions and can be taken out of the integral. This smooth variation is clear for \( \varphi_i, D(\rho_i) \) and \( \mathcal{S}(\rho_i) \) on \( V'_c(r_k) \) and for \( \Phi^{(k)}(r, t) \) and \( \partial_i \Phi^{(k)}(r, t) \) is a consequence of (2.15) and our assumptions on \( \Psi(r, t) \).

One obtains (using \( \cos m_k \varphi_k = \cos \varphi_k, \sin m_k \varphi_k = m_k \sin \varphi_k \) due to \( m_k^2 = 1 \))

\[
\int d\rho_k \left( \cos \varphi_k \right) \left( \frac{1}{1 + \alpha^2} \left[ D'(\rho_k) \hat{\rho}_k + \alpha D(\rho_k) L_k \right] \cdot \hat{r_k} - 2 D(\rho_k) L_k \cdot \nabla \Phi^{(k)} \right) \left( m_k \sin \varphi_k \right)
\]

\[
+ \int d\rho_k \left( \frac{1}{1 + \alpha^2} \left[ -\alpha D'(\rho_k) \hat{\rho}_k + D(\rho_k) L_k \right] \cdot \hat{r_k} + 2 D'(\rho_k) \hat{\rho}_k \cdot \nabla \Phi^{(k)} \right) \left( -m_k \sin \varphi_k \cos \varphi_k \right) = 0.
\]

(2.21)

The integral over \( d\varphi_k \) can now be done. Since

\[
\hat{\rho}_k = (\cos \varphi_k, \sin \varphi_k), \quad \hat{\varphi}_k = (-\sin \varphi_k, \cos \varphi_k),
\]

one has

\[
\int_0^{2\pi} d\varphi_k \left( \begin{array}{c} \cos \varphi_k \\ m_k \sin \varphi_k \end{array} \right) = \left( \begin{array}{c} \pi \\ 0 \end{array} \right)
\]

and other similar results. We use this in (2.21) and in the terms like \( \hat{\rho}_k \cdot \nabla \Phi^{(k)}(r, t) \) we evaluate \( \nabla \Phi^{(k)}(r = r_k) \) before integration in agreement with our assumptions.

This gives

\[
\int_0^{\epsilon} d\rho_k \rho_k \left[ \left( D'(\rho_k) - S_\xi(\rho_k) D(\rho_k) + \frac{D(\rho_k)}{\rho_k} \right) \left( \begin{array}{c} \hat{r}_k \\ -m_k \hat{\varphi}_k \end{array} \right) \right]
\]

\[
+ \left[ -\alpha \frac{D(\rho_k)}{\rho_k} - \alpha D'(\rho_k) - S_\xi(\rho_k) D \right] \left( \begin{array}{c} -m_k \hat{y}_k \\ \hat{x}_k \end{array} \right)
\]

\[
= \int_0^{\epsilon} d\rho_k \rho_k \left[ m_k \left( D'(\rho_k) + \frac{D(\rho_k)}{\rho_k} \right) \left( \begin{array}{c} \partial_i \Phi^{(k)}(r_k) \\ -m_k \partial_i \Phi^{(k)}(r_k) \end{array} \right) - S_\xi(\rho_k) D(\rho_k) \left( \begin{array}{c} \partial_i \Phi^{(k)}(r_k) \\ m_k \partial_i \Phi^{(k)}(r_k) \end{array} \right) \right].
\]

(2.22)

In \( V'_c(r_k) \) one has \( D(\rho_k) \approx \lambda \rho_k \), then

\[
\int_0^{\epsilon} d\rho_k \rho_k D'(\rho_k) - \int_0^{\epsilon} d\rho_k D(\rho_k) = \frac{1}{2} \epsilon \lambda \epsilon^2.
\]

Putting

\[
\int_0^{\epsilon} d\rho_k \rho_k D(\rho_k) S_\xi(\rho_k) = \lambda \epsilon^2 K,
\]

\[
\mathbf{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
\]
we obtain (note $J^2 = -1$)

$$\frac{1}{1 + \alpha^2} (m_k - \alpha J) (m_k + KJ) \begin{pmatrix} \dot{x}_k \\ \dot{y}_k \end{pmatrix} = 2(K - m_k J) \begin{pmatrix} \partial_x \Phi^{(k)}(r_k) \\ \partial_y \Phi^{(k)}(r_k) \end{pmatrix},$$

(2.23)

where we see that $K$ cancels and we have

$$\dot{r}_k = \begin{pmatrix} \dot{x}_k \\ \dot{y}_k \end{pmatrix} = -2J(m_k + \alpha J) \begin{pmatrix} \partial_x \Phi^{(k)}(r_k) \\ \partial_y \Phi^{(k)}(r_k) \end{pmatrix},$$

(2.24)

which can also be written as

$$\frac{dr_k}{dt} = -2m_k \dot{\varphi} \wedge \partial \Phi^{(k)}(r_k) + 2\alpha \frac{\partial \Phi^{(k)}(r_k)}{\partial r_k},$$

(2.25)

which is eq. (12) in ref. [3]. We see then that the motion of a given spiral (here the $k$th spiral) is determined by the phase $\Phi^{(k)}$ seen by this spiral. Indeed, as shown by (2.15b), $\Phi^{(k)}$ is the total phase $\Theta^0$ minus the proper phase of the $k$th spiral ($m_k \varphi_k - S(\rho_k)$). Eq. (2.25) is one of our main results here and shows that $\dot{r}_k$ is the sum of two terms: a Peach–Köhler term proportional to $\dot{\varphi} \wedge \nabla \Phi^{(k)}$ which corresponds to the variational limit ($\alpha = \beta = 0$) and was found by Kawasaki [7] plus a second term proportional to $\nabla \Phi^{(k)}$ which corresponds to the limit in which the complex Ginzburg–Landau equation reduces to the nonlinear Schrödinger equation ($\alpha = \beta \to \infty$) and was found by Fetter [8] (see appendix C). Using (2.16c) one has

$$\frac{dr_k}{dt} = 2 \sum_{i \neq k} \left( \frac{m_i m_k}{r_{ik}} - \alpha S_i(r_{ik}) \right) \dot{r}_{ik} + \alpha \nabla \Psi(r_k) + m_k \dot{\varphi} \wedge \left[ \sum_{i \neq k} \left( \frac{m_i m_k}{r_{ik}} + S_i(r_{ik}) \right) \dot{r}_{ik} - \nabla \Psi(r_k) \right],$$

(2.26)

We proceed now to determine the vector $\chi_{III} \in \text{Ker } \mathcal{L}^\dagger$, which will give us the equation of motion of the new global phase $\Psi(r, t)$. We look for a vector $\chi_{III}$ vanishing in the cores $V_\epsilon(r_j)$, $j = 1, 2, \ldots, N$, and consequently we have to determine this vector only outside the cores, where we can approximate the operator $\mathcal{L}^\dagger$ taking into account the asymptotic behavior of the functions $D(r)$ and $S_i$ for $r \gg \epsilon$. Furthermore in a first approximation we neglect all the gradients. Then in the region $(D_k - V_\epsilon(r_k))$ near the $k$th spiral (but excluding the core $V_\epsilon(r_k)$) we can write

$$\mathcal{L}^\dagger = \begin{pmatrix} -2\gamma D^2_k/(1 + \alpha^2) & -2\nu D^2_k/(1 + \alpha^2) \\ 0 & 0 \end{pmatrix} + \mathcal{O} \left( \frac{1}{\rho_j}, q \nabla, \frac{1}{\rho_j} \nabla \right).$$

(2.27)

We see that

$$\chi_{III} = \begin{pmatrix} \nu \\ -\gamma \end{pmatrix},$$

which in the case of vortices of real Ginzburg–Landau equation reduces to the result obtained in
appendix A (see (A.39)). The solvability condition \( \langle \chi_{\text{III}}, I \rangle = 0 \) gives

\[
\int \, dr \left\{ \beta \, \partial_t R^0 - R^0 \partial_t \theta^0 + \nu R^0 \left[ \left( \nabla \theta^0 \right)^2 - q^2 \right] + \gamma R^0 \nabla^2 \theta^0 - \nu \nabla^2 R^0 + 2 \gamma \nabla R^0 \cdot \nabla \theta^0 \right\} = 0. \tag{2.28}
\]

In order to be able to solve \( \mathcal{L}w = I \) in the same approximation as (2.27) we have to impose that the integrand in (2.28), which is simply \((\nu I_1 - \gamma I_2)\) with \(I_1\) and \(I_2\) given by (2.13), vanishes at each point \(r\). We use now (2.17) and the known behavior of \( S(r) \) and \( D(r) \) for \( r \gg \epsilon \); then in each region \( D_k \) we obtain from (2.28) that

\[
R^0 \partial_t \theta^0 = R^0 \left\{ \nu \left[ \mathbf{L}_k^2 - q^2 + \left( \nabla \Phi^{(k)} \right)^2 + 2 \mathbf{L}_k \cdot \nabla \Phi^{(k)} \right] + \gamma \nabla^2 \Phi^{(k)} - \gamma \nabla_{\rho_k}^2 S(\rho_k) \right\} - \nu \nabla^2 D(\rho_k) - 2 \gamma D'(\rho_k) S_r(\rho_k) + \Theta \left( \frac{q^2}{\rho_j^2}, \frac{q}{\rho_j^3}, \frac{q}{\rho_j^4} \right). \tag{2.29}
\]

This equation of motion can be simplified noting that from (2.7a) and (2.7b) one has

\[
\nu \left( \mathbf{L}_k^2 - q^2 \right) D(\rho_k) - \nu \nabla_{\rho_k}^2 D(\rho_k) - 2 \gamma D'(\rho_k) S_r(\rho_k) + \gamma D \nabla_{\rho_k}^2 S(\rho_k) = 0. \tag{2.30}
\]

Using (2.30) and (2.17c) in (2.29) one obtains

\[
\partial_t \Phi^{(k)} - \mathbf{L}_k \cdot \dot{\mathbf{r}}_k = \gamma \nabla^2 \Phi^{(k)} + \nu \left( \nabla \Phi^{(k)} \right)^2 + 2 \nu \mathbf{L}_k \cdot \nabla \Phi^{(k)}. \tag{2.31}
\]

We can now solve \( \mathcal{L}w = I \) in this approximation (see (2.27)). In each region \( D_k \) one obtains

\[
w = \frac{1 + \alpha^2}{2D_\alpha} \frac{q}{\nu} \left( \sum_j \frac{1}{\rho_j} - \frac{1}{\rho_k} \right) + \tilde{w}, \tag{2.32a}
\]

\[
\tilde{w} = - \frac{1}{2D_\alpha} \left( \left( \nabla \Phi^{(k)} \right)^2 + 2 \mathbf{L}_k \cdot \nabla \Phi^{(k)} + \alpha \nabla^2 \Phi^{(k)} \right). \tag{2.32b}
\]

Since \( w \) is real this correction only changes the amplitude \( R^0 \) of the ansatz (see (2.9)) which becomes (using (2.8a) and (2.10a))

\[
R^0 + w = D(\rho_k) + \tilde{w} + \Theta \left( \frac{1}{\rho_k^2} \right). \tag{2.33}
\]

Using the modified ansatz we can calculate corrections to the phase equation (2.31) and we obtain

\[
\partial_t \Phi^{(k)} - \mathbf{L}_k \cdot \dot{\mathbf{r}}_k = \gamma \nabla^2 \Phi^{(k)} + \nu \left( \nabla \Phi^{(k)} \right)^2 + 2 \nu \mathbf{L}_k \cdot \nabla \Phi^{(k)}
- \frac{2(1 + \beta^2)}{D_\alpha^2} \left( \mathbf{L}_k \cdot \nabla \right)^2 \Phi^{(k)} - \frac{\alpha^2(1 + \beta^2)}{2D_\alpha^2} \nabla^4 \Phi^{(k)} + \Theta (\nabla^3). \tag{2.34}
\]
From (2.34) and the expression (2.15c) of $\Phi^{(k)}$ in terms of the new global phase $\Psi$ we obtain

$$
\partial_t \Psi = \gamma \nabla^2 \Psi + \nu (\nabla \Psi)^2 + 2\nu (L_k + \Sigma_{l_i}) \cdot \nabla \Psi - \frac{2(1 + \beta^2)}{D_x^2} (L_k \cdot \nabla)^2 \Psi
$$

$$
+ 2\nu L_k \cdot \Sigma_{l_i} + \nu (\Sigma_{l_i})^2 + L_k \cdot \dot{r}_k + \Sigma_{l_i} \cdot \dot{r}_i,
$$

(2.35)

where

$$
l_i = \frac{m_i}{\rho_i} \phi_i - \psi_i(\rho_i) \dot{r}_i
$$

(2.36)

and $\Sigma'$ stands for sum over $i \neq k$.

We shall approach now the problem of the phase equation by another method based on adiabatic elimination. Near $r = r_k$, i.e. near the $k$th spiral, we can write the solution of (2.2) as

$$
A = \left[ D(\rho_k) + \rho \right] \exp \left[ i (\omega t + m_k \varphi_k - S(\rho_k) + \phi) \right],
$$

(2.37)

i.e. in the form of a single spiral perturbed by $(\rho(r, t), \phi(r, t))$. Adiabatic elimination of $\rho$ results in an equation for the phase $\phi$ which is (see appendix B)

$$
\partial_t \phi = \gamma \nabla^2 \phi + \nu (\nabla \phi)^2 + 2\nu L_k \cdot \nabla \phi - \frac{2(1 + \beta^2)}{D_x^2} (L_k \cdot \nabla)^2 \phi - \frac{\alpha^2(1 + \beta^2)}{2D_x^2} \nabla^4 \phi + \mathcal{E}(\nabla^3).
$$

(2.38)

This equation coincides with (2.34) when one considers the $k$th spiral at rest and reduces for $L_k = -k$ to the equation for a phase perturbation to the periodic pattern solution

$$
A = \sqrt{\mu} - k^2 \exp \left[ i (\omega t - k \cdot r) \right], \quad \omega = -\nu(\mu - k^2)
$$

of the Ginzburg–Landau equation. The linear part of (2.38) is the linear stability equation for the phase in the external region of a single spiral solution (see appendix B). Using (2.8b) one has

$$
L_k \cdot \nabla = -\left( q + \frac{\gamma}{2\nu \rho_k} \right) \frac{\partial}{\partial \rho_k} + \mathcal{E}(\rho_k^{-2}),
$$

$$
(L_k \cdot \nabla)^2 = \left( q^2 + \frac{\gamma q}{\nu \rho_k} \right) \frac{\partial^2}{\partial \rho_k^2} + \mathcal{E}(\rho_k^{-2})
$$

and from (2.38) we obtain

$$
\partial_t \phi = -2q\nu \frac{\partial \phi}{\partial \rho_k} + l(q^2) \frac{\partial^2 \phi}{\partial \rho_k^2} + \nu \left( \frac{\partial \phi}{\partial \rho_k} \right)^2
$$

(2.39)

with

$$
l(q^2) = \gamma - \frac{2(1 + \beta^2)}{D_x^2} \left( q^2 + \frac{\gamma q}{\nu \rho_k} \right) \approx \gamma - \frac{2(1 + \beta^2)q^2}{D_x^2}.
$$

(2.40)
In agreement with (2.15b) and (2.37) we identify \( \phi \) and \( \Phi^{(k)} \). Replacing \( \Phi^{(k)} \) by (2.15c) and taking into account that

\[
\frac{\partial^2 \Phi^{(k)}}{\partial \rho_k^2} = \frac{\partial^2 \Psi}{\partial \rho_k^2} + \epsilon \left( \frac{1}{\rho_j^2} \right)
\]

we obtain in leading order from (2.38) the phase equation in the form

\[
\partial_t \Psi = -2q \nu \frac{\partial \Psi}{\partial \rho_k} + l(q^2) \frac{\partial^2 \Psi}{\partial \rho_k^2} + \nu \left( \frac{\partial \Psi}{\partial \rho_k} \right)^2 + \ldots .
\]  

(2.41)

3. Some consequences of the equations of motion for the spirals and the phase

We have already discussed in ref. [3] the systems spiral–spiral and spiral–antispiral when the new global phase \( \Psi \) can be neglected. The results were in very good agreement with numerical results [5]. We shall discuss now the stability of the new global phase from eq. (2.41). This equation shows that the phase \( \Psi \) will be unstable when

\[
l(q^2) = \gamma - \frac{2(1 + \beta^2)q^2}{\mu - q^2} < 0,
\]

(3.1)

and in this situation strong gradients will arise which produce new spirals and maintain the state dominated by defects. The inequality (2.36) is satisfied by \( \gamma < 0 \) but also by \( \gamma > 0 \) till a critical value \( \gamma_c \),

\[
\gamma_c = \frac{2(1 + \beta^2)q^2}{\mu - q^2}.
\]

This value can be calculated once \( q \) is known. We determine \( q \) for \( \gamma = 1 + \alpha \beta > 0 \) making in (2.2) the change of variables

\[
A(\mathbf{r}, t) = \xi \sqrt{\frac{1 + \alpha^2}{\gamma}} B(\xi \mathbf{r}) \exp(i \omega_1 t),
\]

(3.2)

which gives

\[
\frac{i \omega_1}{1 + \alpha^2} B + \frac{\alpha \omega_1}{1 + \alpha^2} B = \mu B + \xi^2 \left[ \nabla^2 B - (1 + i s) |B|^2 B \right]
\]

(3.3)

with

\[
s = \frac{\beta - \alpha}{1 + \alpha \beta} = \frac{\nu}{\gamma}.
\]

If we choose

\[
\xi^2 = \mu - \frac{\alpha \omega_1}{1 + \alpha^2},
\]

(3.4a)
and
\[
\frac{\omega_1}{1 + \alpha^2} = -\xi^2 s(1 - q_1^2),
\]  
(3.4b)

eq. (3.3) becomes
\[
-is(1 - q_1^2)B = B + \nabla^2 B - (1 + is) |B|^2 B.
\]  
(3.5)

This is the equation studied by Hagan [6], who finds a relation \( q_1 = q_1(s) \) and the asymptotic behavior
\[
B(|r| \to \infty) = \sqrt{1 - q_1^2} \exp(isq_1 r).
\]  
(3.6)

Comparing this with the known asymptotic behavior of \( A \) through (3.2) we obtain
\[
q = \xi q_1 = \frac{\sqrt{\mu}}{\sqrt{1 - \alpha s(1 - q_1^2)}}, \quad q_1 = q_1(s = \nu/\gamma),
\]  
(3.7)

where \( \xi \) was obtained from (3.4). The function \( q_1 = q_1(s) \) is given in ref. [6] and then (3.7) determines \( q \) for \( \gamma > 0 \), using this value we can calculate \( \gamma_s \). There are some situations where \( \gamma \) is not a good parameter, for instance if \( \alpha = 0 \) one has \( \gamma = 1 \) and \( \beta \) is now the good variable. For these cases we write (using (3.7)) \( l(q^2) \) for \( \gamma > 0 \) in the form
\[
l(q^2) = \gamma \left( 1 - \frac{2(1 + \beta^2)}{1 + \alpha^2} \frac{q^2}{1 - q_1^2} \right).
\]  
(3.8)

The condition \( l(q^2) < 0 \) is now written
\[
q_1^2 > \frac{1 + \alpha^2}{3 + 2\beta^2 + \alpha^2}, \quad q_1 = q_1(\frac{\beta - \alpha}{1 + \alpha \beta}).
\]  
(3.9)

We have represented graphically (3.9) in fig. 2, where we have drawn in the \((\alpha, \beta)\) plane the curves \( q_1^2 = (1 + \alpha^2)/(3 + 2\beta^2 + \alpha^2) \) and \( 1 + \alpha \beta = 0 \) using the curve \( q_1 = q_1(s) \) provided by Hagan [6].

We have now the following picture. If we take an initial condition of (2.2) without defects, for example the perfect pattern \( A = \sqrt{\mu} \exp(-i\nu t), \) then the phase \( \phi \) defined by \( A = \sqrt{\mu} + \rho \exp[-i(\nu \mu t + \phi)] \) satisfied the Kuramoto–Sivashinsky equation (see appendix B)
\[
\partial_t \phi = \gamma \nabla^2 \phi - \frac{(1 + \beta^2)\alpha^2}{2\mu} \nabla^4 \phi + \nu(\nabla \phi)^2.
\]  
(3.10)

If \( \gamma > 0 \), no defects appear and \( \phi \) is stable. When \( \gamma \) becomes negative, \( \phi \) becomes unstable, strong gradients arise and finally spirals appear [2]: we are then in the state dominated by defects described by our ansatz, eq. (3.10) is not valid anymore and is replaced by eq. (2.41) for the new global phase \( \Psi \). If we are now in this regime dominated by spirals and we start to increase \( \gamma \) we can see from (2.41) that the
phase continues to be unstable when we cross the previous critical point \( \gamma = 0 \) and this till \( \gamma_c > 0 \) determined by \( l(q^2) = 0 \) (see (3.1)). After crossing \( \gamma_c \) the phase \( \Psi \) becomes stable and spirals are not created anymore. This mechanism provides then a possible explanation to the hysteresis loop observed in ref. [2]. In that experience one has \( \mu = 1, \beta = -2, \) and \( \alpha \) is variable. We have calculated \( \gamma_c \) using (3.11) and obtained the approximate result (the curve \( q_1 = q_1(s) \) is given only graphically by Hagan) \( \gamma_c \approx 1.6 \), which is to be compared with the experimental value \( \gamma_{c,\text{exp}} \approx 1.0 \).

In many cases it is convenient to perform the analysis of \( l(q^2) < 0 \) directly in the \((\alpha, \beta)\) plane as already discussed above. In fact the experience in ref. [2] corresponds in fig. 2 to moving in the horizontal line \( \beta = -2 \) is fixed) first to the right to the region \( 1 + \alpha \beta < 0 \) and then to coming back to the hysteresis region between the two curves \( \gamma = 0 \) and \( l(q^2) = 0 \).

We think therefore that it is the phase instability which is at the origin of the spontaneous creation of defects. Another confirmation of this fact can be found in the numerical experiment in ref. [8] (for instance, p. 139), where the initial condition is an isolated spiral whose phase stability condition in the outside region is again (3.1) (see appendix B). In this experiment one again observes spontaneous creation of spirals when the values of the parameters are such that \( l(q^2) < 0 \). Let us finally recall that in
the numerical experiment in ref. [5] it can easily be checked that one is in the region of phase stability according to (3.1), i.e. \( l(q^2) > 0 \), and indeed no creation of defects is observed.

In summary, we can say concerning this point that the cause of the spontaneous creation of defects is phase instability. If this happens, spirals will appear and the system goes to the regime dominated by defects and these defects will remain as long as the new global phase is unstable.

We now go back to eqs. (2.26) and replace there the asymptotic behavior (2.8b) of \( S_r(r_{ik}) \):

\[
\frac{dr_k}{dt} = 2 \left[ \sum_{i \neq k} \left( m_i m_k - \frac{\alpha \gamma}{2 \nu} \right) \frac{\hat{r}_{ik}}{r_{ik}} + m_k \sum_{i \neq k} \left( \alpha m_i m_k + \frac{\gamma}{2 \nu} \right) \hat{z} \cdot \frac{\hat{r}_{ik}}{r_{ik}} + m_k \hat{z} \cdot \nabla \Psi(r_k) + \alpha \nabla \Psi(r_k) \right].
\]

(3.11)

Then the first term in the right-hand side of (3.11) tells us that two spirals will attract themselves if

\[
m_i m_k - \frac{\alpha \gamma}{2 \nu} < 0
\]

and tend to separate in the opposite case. If two spirals with different charge \( m_i m_k = -1 \) get very near they can annihilate when their distance is of the order \( \epsilon \) (the dimension of the core). Our equations are not valid in this region but conservation of topological charge allows annihilation and that is what is observed numerically [2, 9].

If the two spirals have equal charges \( m_i m_k = 1 \) and if \( 1 - \alpha \gamma / 2 \nu < 0 \) we shall have attraction. However, conservation of charge forbids now annihilation of the spirals and since we are now in the region where our equations are not valid we can assume that near the core is a barrier, i.e. a repulsive interaction. In this situation a stable bound state of two spirals may arise which will have topological charge \( \pm 2 \). In most numerical experiments one is in the region \( \alpha \gamma / 2 \nu < 1 \) and the described effect cannot be observed. For the experience in ref. [10] one has \( \mu = 1, \beta = -2 \) and \( \alpha \) is variable. The critical value of \( \alpha \) is \( \alpha_c = 2.35 \) and it would be of interest to have detailed information for \( \alpha > \alpha_c \). In fact, in ref. [10] the number of defects seems to decrease for \( \alpha > \alpha_c \) and this could be related to the coalescence of two spirals with \( m_i m_k = 1 \) in a defect with charge \( \pm 2 \).

Appendix A

The Ginzburg–Landau equation with real coefficients is \( (A \in \mathbb{C}) \)

\[
\partial_t A = \mu A + \nabla^2 A - A|A|^2. \quad (A.1)
\]

The vortex solution is of the form \( A(r) = D(r)e^{im\phi}, m \in \mathbb{Z} \), and one obtains from (A.1) the equation

\[
\mu D + \nabla^2 D - \frac{m^2}{r^2} D - D^3 = 0. \quad (A.2)
\]

From (A.2) one obtains the asymptotic behavior

\[
D(r) = \lambda_m r^{|m|}, \quad r \to 0 \quad (A.3)
\]

\[
D(r) = \sqrt{\mu} - \frac{m^2}{2\sqrt{\mu} r^2} + \mathcal{O}(r^{-3}), \quad r \to \infty. \quad (A.4)
\]
We introduce the vector notation
\[
\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = R \begin{pmatrix} \cos \Theta \\ \sin \Theta \end{pmatrix}
\]
for the field \( A = R e^{i \Theta} \). Eq. (A.1) becomes
\[
\partial_t \phi = (\mu + \nabla^2) \phi - \phi^2 \phi = F(\phi)
\]
(A.5)
with \( \phi^2 = \phi_1^2 + \phi_2^2 \). We now look for a solution dominated by well separated vortices (a diluted gas of vortices). We choose the following ansatz to represent a solution with \( N \) vortices at \( r_1, r_2, \ldots, r_N \) (\( \hat{\phi}_i \) and \( \hat{\phi}_i \) are as in section 2 for the spirals)
\[
\phi = \phi^0(r_i(t), \Psi(r, t), r) + w(r_i(t), \Psi(r, t), r),
\]
(A.6)
\[
\phi^0 = R^0 \begin{pmatrix} \cos \Theta^0 \\ \sin \Theta^0 \end{pmatrix},
\]
(A.7)
\[
R^0 = \sum_{i=1}^{N} \left[ D(\rho_i) - \sqrt{\mu} \right] + \sqrt{\mu},
\]
(A.8)
\[
\Theta^0 = \sum_{i=1}^{N} m_i \varphi_i(r) + \Psi(r, t).
\]
(A.9)
Here \( w \) is a small correction and \( \Psi(r, t) \) is a new global phase which has to be introduced to satisfy the solvability conditions as we shall see. The ansatz is chosen such that \( \phi^0 \) behaves locally as a vortex near each point \( r_k \). One has
\[
\phi^0(r \approx r_k) = D(\rho_k) \begin{pmatrix} \cos \left[ m_k \varphi_k + \alpha^k(r_k) \right] \\ \sin \left[ m_k \varphi_k + \alpha^k(r_k) \right] \end{pmatrix} = \Xi^{(k)}(\rho_k)
\]
(A.10)
with
\[
\alpha^k(r_k) = \sum_{i \neq k} m_i \varphi_i(r_k) + \Psi(r_k, t).
\]
(A.11)
The function \( \alpha^k(r_k) \) varies slowly in the core \( \psi_k(r_k) \) and represents the value of the total phase in \( r \) after subtracting the contribution of the \( k \)th vortex. Introducing (A.6) in (A.1) and keeping only linear terms in \( w \) we obtain (we are assuming \( w, \dot{r}_k \) and \( \Psi \) are small quantities of the same order)
\[
\partial_t \phi^0 = F(\phi^0) + \mathcal{L}(\phi^0) w.
\]
(A.12)
Here \( \mathcal{L} \) is a symmetric matrix defined by
\[
\mathcal{L}_{ik}(\phi^0) = \frac{\delta F_i}{\delta \phi_k}(\phi^0).
\]
(A.13)
Eq. (A.12), considered as an equation for \( w \), implies the following solvability condition:

\[
\langle \partial_t \phi^0 - F(\phi^0), X \rangle = 0
\]  

(A.14)

with the scalar product \( \langle Y(r), Z(r) \rangle = \int dr (Y_1 Z_1 + Y_2 Z_2) \) and \( X \in \text{ker } \mathcal{L}^\dagger = \text{ker } \mathcal{L} \) since \( \mathcal{L}^\dagger = \mathcal{L} \) due to (A.13), i.e. \( \mathcal{L}X = 0 \). The elements of \( \text{ker } \mathcal{L} \) are determined by the properties of invariance of (A.1) under continuous transformations (Goldstone theorem). These transformations are here space translations and \( A \rightarrow A e^{ia} \) (the rotational invariance is the same as this since if we rotate an isolated vortex in an angle \( \delta \) then the phase varies in \( m_k \delta \)). We recall that near the \( k \)th vortex one has \( \phi^0(r = r_k) = \Xi^{(k)}(\rho_k) \), but this function is an exact solution of (A1)-(A5) and we have \( F(\Xi^{(k)}) = 0 \). As a consequence of translational invariance the center \( r_k \) of the vortex can be displaced and this implies taking derivative of \( F(\Xi^{(k)}) = 0 \) with respect to \( r_k = (x_k, y_k) \) that

\[
\mathcal{L}'(\Xi^{(k)}) \frac{\partial \Xi^{(k)}}{\partial x_k} = 0,
\]

(A.15a)

\[
\mathcal{L}'(\Xi^{(k)}) \frac{\partial \Xi^{(k)}}{\partial y_k} = 0.
\]

(A.15b)

We note that \( \mathcal{L} \) defined by (A.13) is evaluated here at \( \Xi^{(k)} \) and not in \( \phi^0 \), but in the neighborhood of \( r_k \) these two functions coincide and consequently \( \partial \Xi^{(k)}/\partial x_k \) and \( \partial \Xi^{(k)}/\partial y_k \) are in \( \text{ker } \mathcal{L} \) in the core \( V_\epsilon(r_k) \), but due to the asymptotic behavior of \( \Xi^{(k)}(\rho_k) \) for \( |\rho_k| \gg \epsilon \) the derivatives tend to zero in this region and consequently one has everywhere

\[
\mathcal{L}'(\phi^0) \frac{\partial \Xi^{(k)}}{\partial x_k} = 0,
\]

(A.16a)

\[
\mathcal{L}'(\phi^0) \frac{\partial \Xi^{(k)}}{\partial y_k} = 0.
\]

(A.16b)

We then have here \( 2N \) elements of \( \text{ker } \mathcal{L} \) \( (k = 1, 2, \ldots, N) \) and we expect that they will give through the solvability condition (A.14) the equations of motion for \( r_k = (x_k, y_k) \). From now on we restrict ourselves to \( |m_k| = 1 \). An elementary calculation gives that in \( V_\epsilon(r_k) \), i.e. for \( \rho_k < \epsilon \), one has

\[
\frac{\partial \Xi^{(k)}}{\partial x_k} = -\lambda \cos \alpha^{(k)}, \quad \frac{\partial \Xi^{(k)}}{\partial y_k} = -\lambda \sin \alpha^{(k)},
\]

(A.17)

\[
\frac{\partial \Xi^{(k)}}{\partial y_k} = \lambda m_k \sin \alpha^{(k)}, \quad \frac{\partial \Xi^{(k)}}{\partial x_k} = -\lambda m_k \cos \alpha^{(k)}.
\]

(A.18)

All these derivatives vanish for \( \rho_k > \epsilon \) and \( \lambda \) is the coefficient in (A.3). In (A.17)-(A.18) we have neglected terms which are \( O(\rho_k) \), which is a consistent approximation with what follows. A last element of \( \text{ker } \mathcal{L} \) is associated to the phase invariance \( A \rightarrow A e^{ia} \). This element will give a solvability condition (A.14), which requires to be satisfied by the introduction of the phase \( \Psi(r, t) \) we have included in the ansatz (A.6). We remark that in the intermediate regions outside the cores \( V_\epsilon(r_k) \) one has \( F(\phi^0) = \)
(1/\rho_k^2), i.e. \( \phi^0 \) is a solution there up to an arbitrary phase \( \alpha \), taking the derivative with respect to \( \alpha \):

\[
\mathcal{L}(\phi^0) \frac{\partial \phi^0}{\partial \alpha} = 0, \quad \frac{\partial \phi^0}{\partial \alpha} = \begin{pmatrix} -\phi_2^0 \\ \phi_1^0 \end{pmatrix}
\]  
(A.19)

and \( \partial \phi^0 / \partial \alpha \) is the last element of ker \( \mathcal{L} \). If \( \chi = (\chi_1, \chi_2) \) is one of the elements (A.17), (A.18), (A.19) of ker \( \mathcal{L} \) the solvability condition (A.14) takes the form

\[
\int \, dr \left( \sum_i \hat{x}_i \frac{\partial \phi_1^0}{\partial \hat{x}_i} + \sum_i \hat{y}_i \frac{\partial \phi_1^0}{\partial \hat{y}_i} - \Psi \phi_2^0 - F_1(\phi^0) \right) \chi_1 \\
+ \int \, dr \left( \sum_i \hat{x}_i \frac{\partial \phi_2^0}{\partial \hat{x}_i} + \sum_i \hat{y}_i \frac{\partial \phi_2^0}{\partial \hat{y}_i} + \Psi \phi_1^0 - F_2(\phi^0) \right) \chi_2 = 0.
\]  
(A.20)

We replace first \( \chi \) by \( \partial \Xi^{(k)} / \partial \hat{x}_k \) as given by (A.17); then the region of integration in (A.20) reduces to the core \( V'(r_k) \) and we can replace there \( \phi^0(r) \) by \( \Xi^{(k)}(\rho_k) \). Moreover \( \partial \Xi^{(k)} / \partial \hat{x}_i = \partial \Xi^{(k)} / \partial \hat{y}_i = 0 \) for \( i \neq k \) and the sum in (A.20) reduces to one term. Taking into account that \( \Psi(r, t) \) is approximately constant in \( V'(r, t) \) we replace \( \Psi(r, t) \) by \( \Psi(r_k, t) \), and the corresponding terms in (A.20) vanish since (\( dr = d\rho_k \))

\[
\int \, d\rho_k \phi^0 = \int_0^\pi \rho_k \, d\rho_k \, D(\rho_k) \int_0^{2\pi} \, d\varphi_k \left( \cos(m_k \varphi_k + \alpha^k) \right) = 0.
\]  
(A.21)

One finally obtains

\[
\lambda^2 \pi \varepsilon^2 \hat{x}_k = -\lambda \left( \cos(\alpha^k) \int_{V'(k)} \, d\rho_k F_1(\phi^0) + \sin(\alpha^k) \int_{V'(k)} \, d\rho_k F_2(\phi^0) \right).
\]  
(A.22)

A similar calculation using (A.18) gives

\[
\lambda^2 \pi \varepsilon^2 \hat{y}_k = \lambda m_k \left( \sin(\alpha^k) \int_{V'(k)} \, d\rho_k F_1(\phi^0) - \cos(\alpha^k) \int_{V'(k)} \, d\rho_k F_2(\phi^0) \right).
\]  
(A.23)

One has

\[
F(\phi^0) = (\mu - (R^0)^2) \phi^0 + \nabla^2 \phi^0
\]

and a similar calculation to (A.21) shows that only the \( \nabla^2 \) term survives. We have (using again (A.21) with \( D(\rho_k) \) replaced by any function of \( \rho_k \))

\[
\int \, d\rho_k \nabla^2 \phi_1^0 = -2 \int \, d\rho_k \sin \theta^0 \nabla R^0 \cdot \nabla \theta^0 - \int \, d\rho_k \cos \theta^0 R^0 (\nabla \theta^0)^2,
\]  
(A.24)

\[
\int \, d\rho_k \nabla^2 \phi_2^0 = 2 \int \, d\rho_k \cos \theta^0 \nabla R^0 \cdot \nabla \theta^0 - \int \, d\rho_k \sin \theta^0 R^0 (\nabla \theta^0)^2.
\]  
(A.25)
Using (A.8), (A.9) one obtains

\[ \int d\rho_k \nabla R^0 \cdot \nabla \vartheta^0 \left( \frac{\cos \varphi_k}{\sin \varphi_k} \right) = \frac{\pi e^2}{2} \sum_{i \neq k} \frac{m_i}{r_{ik}} \left( \frac{-\sin \varphi_{ik}}{\cos \varphi_{ik}} \right) + \left( \frac{\partial \vartheta^0}{\partial x} \right)_{r_k}. \]  

(A.26)

\[ \int d\rho_k R^0 (\nabla \vartheta^0)^2 \left( \frac{\cos \varphi_k}{\sin \varphi_k} \right) = \pi e^2 \sum_{i \neq k} \frac{m_i m_k}{r_{ik}} \left( \frac{\cos \varphi_{ik}}{\sin \varphi_{ik}} \right) - m_k \left( \frac{-\partial \vartheta^0}{\partial y} \right)_{r_k}. \]  

(A.27)

where \( r_{ik} = r_k - r_i, \ r_k = |r_k - r_i|, \ \cos \varphi_{ik} = (x_k - x_i)/r_{ik}, \ \sin \varphi_{ik} = (y_k - y_i)/r_{ik}. \)

Putting all this together we obtain

\[ \frac{d \rho_k}{dt} = 2 \sum_{i \neq k} \frac{m_i m_k}{r_{ik}} \hat{r}_{ik} + 2m_k \hat{z} \wedge \nabla \vartheta^0 (r_k), \]  

(A.28)

where \( \hat{r}_{ik} \) is the unitary vector along \( r_{ik} \) and \( \hat{z} \) a unitary vector orthogonal to the plane of the system. In this calculation we have neglected terms \( \mathcal{O}(r_{ik}^{-2}) \). We obtain now the equation for the phase \( \Psi \) from the solvability condition with (A.19). If we neglect the motion of the vortices, i.e. we do not consider the terms with \( (\dot{x}_i, \dot{y}_i) \) in (A.20), we obtain (using \( \nabla^2 \vartheta^0 = \nabla^2 \Psi^0 \))

\[ \int dr (R^0)^2 \left( \frac{\partial \vartheta(r, t)}{\partial \vartheta(r, t)} - \nabla^2 \vartheta(r, t) - 2 \frac{1}{R^0} \nabla R^0 \cdot \nabla \vartheta^0 \right) = 0. \]  

(A.29)

This solvability condition is satisfied if we require the integrand in (A.29) to vanish. Leaving aside the term with \( \nabla R^0 \) of order \( \mathcal{O}(\rho_k^{-3}) \) we obtain the phase equation

\[ \frac{\partial}{\partial t} \vartheta(r, t) = \nabla^2 \vartheta(r, t). \]

We then have phase stability here, and from (A.28) the equation of motion for the vortices in this regime reduces to

\[ \dot{r}_k = 2 \sum_{i \neq k} \frac{m_i m_k}{r_{ik}} \hat{r}_{ik}, \]  

(A.30)

which is a two-dimensional Coulomb gas which exhibits critical behavior in the presence of termal noise [11]. To see this, we define

\[ H = -2 \sum_{i \neq k} m_i m_k \ln \left( \frac{r_{ik}}{\xi} \right), \]  

(A.31)

where \( \xi \) is a constant with dimension of length, then

\[ \frac{dr_k}{dt} = -\frac{1}{2} \frac{\partial H}{\partial r_k}. \]  

(A.32)

We add to the system a \( \delta \)-correlated white noise (without noise the solution of the Ginzburg–Landau equation has no defects):

\[ \frac{dr_k}{dt} = -\frac{1}{2} \frac{\partial H}{\partial r_k} + \sqrt{T} \xi_k(t), \]  

(A.33)
where $T$ is a constant proportional to the temperature and $\xi_k(t)$ is a Gaussian process satisfying
\[
\langle \xi_k(t) \rangle = 0 \quad \text{and} \quad \langle \xi_k(t) \xi_k^b(t') \rangle = \delta_{kk} \delta^{ab} \delta(t-t'). \tag{A.34}
\]

From the Langevin equations (A.33), (A.34) we obtain the corresponding Fokker–Planck equation which admits as solution a stationary probability [12] $p_{\text{st}}$ proportional to $\exp(-H/T)$, the Gibbs factor. In order to obtain this probability we have considered a master equation with only drift and diffusion terms. On the other hand if we consider nucleation and annihilation of vortices we obtain [13] that in the stationary state the probability $p_{NN}$ of having $N$ vortices and $N$ antivortices is proportional to $r^N$, where $r$ is the quotient between the rates of nucleation and annihilation and has dimension inversely proportional to the square of the volume. Putting $r = r'/\xi^4$, where $\xi$ is a length and assuming independence of nucleation–annihilation and drift–diffusion effects one obtains the complete probability distribution [13]
\[
p_{\text{st}} = \frac{1}{Z} \frac{1}{(\xi^2)^{2N}} \exp\left(-\frac{1}{T} \left[ h - (T \ln r') N \right] \right). \tag{A.35}
\]

Here we identify $T \ln r'$ as a "chemical potential" $\tilde{\mu}$ and $Z$ is a normalization function, the partition function, with the value
\[
Z = \sum_{N=0}^{\infty} \frac{1}{(N!)^2} \frac{1}{(\xi^2)^{2N}} \int dr_1 \ldots dr_N \exp\left(-\frac{1}{T} \left( H - \tilde{\mu} N \right) \right), \tag{A.36}
\]

which is the starting point of the work of Kosterlitz and Thouless (1973 and 1974) [12], who found critical behavior at $T = T_c \approx 1$ (the exact value depends on the value of $\tilde{\mu}$).

Let us finally remark that the elements of $\ker \mathcal{L}$ determined in section 2 for the complex Ginzburg–Landau equation (2.2) reduce when $\alpha = \beta = 0$ to the vectors given here (formulas (A.17) to (A.19)). In order to compare we remark that the small perturbation to the ansatz is not parametrized in the same way in both cases. In this appendix we have $A = R^0 \exp(i\Theta^0) + w$ and in section 2 $A = (R^0 + w') \exp(i\Theta^0)$. This gives the relation
\[
\begin{bmatrix}
\text{Re } w \\
\text{Im } w
\end{bmatrix} = \mathbf{B} \begin{bmatrix}
\text{Re } w' \\
\text{Im } w'
\end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix}
\cos \Theta^0 & -\sin \Theta^0 \\
\sin \Theta^0 & \cos \Theta^0
\end{bmatrix}. \tag{A.37}
\]

If we consider the vector $\chi_{\text{III}}$ (see after (2.27)) we obtain
\[
\mathbf{B} \begin{bmatrix}
\nu \\
-\gamma
\end{bmatrix} = \begin{bmatrix}
\nu \cos \Theta^0 - \gamma \sin \Theta^0 \\
\nu \sin \Theta^0 + \gamma \cos \Theta^0
\end{bmatrix}, \tag{A.38}
\]

which is (A.19) for $\nu = \beta - \alpha = 0$, since $\phi^0_1 = R^0 \cos \Theta^0 \approx \sqrt{\mu} \cos \Theta^0$, $\phi^0_2 = R^0 \sin \Theta^0 \approx \sqrt{\mu} \sin \Theta^0$ outside the cores. From (2.19) we obtain
\[
\mathbf{B} \chi^{(k)}_1 = \begin{bmatrix}
\cos(\Theta^0 - m_k \varphi_k) \\
\sin(\Theta^0 - m_k \varphi_k)
\end{bmatrix}, \quad \mathbf{B} \chi^{(k)}_{\text{II}} = \begin{bmatrix}
\sin(m_k \varphi_k - \Theta^0) \\
\cos(\Theta^0 - m_k \varphi_k)
\end{bmatrix}. \tag{A.39}
\]
For vortices $\Theta^0 = m_k \varphi_k + \alpha_k$ and these vectors reduce to (A.17) and (A.18). We then see here the relation of the elements in the ker $\varphi^+$ determined in section 2 with Goldstone's theorem.

Appendix B

In this appendix we study the phase stability of a one-armed spiral wave. The problem of the stability of a spiral is complicated [6] and it is not yet settled. But we are interested in the stability in the external region, i.e. where the derivatives of $D(r)$ and $S_r$ can be neglected. We consider a perturbation of a spiral solution of the Ginzburg–Landau equation

$$\frac{\partial A}{\partial t} = (1 + i\alpha) \left( \mu A + \nabla^2 A - \frac{\gamma + i\nu}{1 + \alpha^2} |A|^2 A \right)$$

and we write $A$ in the form

$$A = [D(r) + \rho(t)] \exp\{i[\omega t + m\varphi - S(r) + \phi] \},$$

where $D(r)$ and $S(r)$ satisfy the relations (2.4) and (2.7) of the spiral solution and $\rho(r, t), \phi(r, t)$ are the perturbations of $D(r)$ and $S(r)$.

Introducing (B.2) in (B.1) one obtains

$$\frac{\partial \rho}{\partial t} + i(D_m + \rho) \frac{\partial \phi}{\partial t} = (1 + i\alpha) \left( \frac{\gamma + i\nu}{1 + \alpha^2} (\rho - 3D_m\rho^2 + \rho^3) + \nabla^2 \rho - D_{\phi}(\nabla \phi)^2 - 2D_{\phi}L \cdot \nabla \phi \\
- 2\rho L \cdot \nabla \phi - \rho(\nabla \phi)^2 + 2i\nabla \rho \cdot (L + \nabla \phi) + i\rho \nabla^2 \phi + iD_{\phi} \nabla^2 \phi \right),$$

here

$$L = \frac{m}{r} \dot{\varphi} - S_r(r) \hat{r},$$

and we have used the equation satisfied by the single spiral solution (from (2.7)):

$$\frac{\gamma + i\nu}{1 + \alpha^2} (D_m^2 - D^2)D + \nabla^2 D - D(L^2 - q^2) + 2iL \cdot \nabla D - iD \nabla^2 S = 0$$

and neglected some terms of order of $1/r$, because we replace $D$ by $D_m$ and $L^2$ by $q^2$. Note that the term $L$ does not commute with $\nabla$.

If we study only the linear part of (B.3) we note that $\rho$ is linearly stable (i.e. the real part of its eigenvalue is negative) and consequently is damped. We assume that $\rho$ is only a function of the gradients of $\phi$, i.e. $\rho = G(\nabla \phi)$ and also that $\phi$ has an autonomous dynamics, i.e. $\partial_t \phi = \Phi(\nabla \phi)$ [9, 14].

Based on (B.3) we write that the functions $G$ and $F$ are a combination of $L \cdot \nabla \phi, \nabla^2 \phi, (L \cdot \nabla)^2 \phi, (\nabla \phi)^2, \ldots$

$$\rho = v_1(L \cdot \nabla) \phi + v_2 \nabla^2 \phi + v_3(\nabla \phi)^2 + v_4(L \cdot \nabla)^2 \phi + v_5(L \cdot \nabla)^2 \phi + \ldots$$

$$+ v_7(L \cdot \nabla)^2 \phi + v_8 \nabla^2 (L \cdot \nabla) \phi + v_9(L \cdot \nabla)^3 \phi$$

(B.6)
and

\[
\partial_t \phi = u_1 (L \cdot \nabla) \phi + u_2 \nabla^2 \phi + u_3 (\nabla \phi)^2 + u_4 (L \cdot \nabla \phi)^2 + u_5 (L \cdot \nabla \phi)^3 + u_6 \nabla^4 \phi \\
+ u_7 (L \cdot \nabla) \nabla^2 \phi + u_8 \nabla^2 (L \cdot \nabla) \phi + u_9 (L \cdot \nabla)^3 \phi,
\]

where the \(\{u_i, v_i\}\) are quantities to be determined. Using (B.6) and (B.7) one has

\[
\partial_t \rho = \frac{\partial G}{\partial \phi} \partial_t \phi = u_1 v_1 (L \cdot \nabla) \phi + u_2 v_2 (L \cdot \nabla) \nabla^2 \phi + u_3 v_3 \nabla^2 (L \cdot \nabla) \phi
\]

\[
+ (u_4 v_1 + u_4 v_4) (L \cdot \nabla)^3 \phi + u_2 v_2 \nabla^4 \phi,
\]

(\text{B.7})

\[
\nabla^2 \rho = v_1 \nabla (L \cdot \nabla) \phi + v_2 \nabla^2 \nabla^2 \phi + v_3 \nabla (L \cdot \nabla)^2 \phi,
\]

(\text{B.8a})

\[
\nabla^2 \rho = v_1 \nabla^2 (L \cdot \nabla) \phi + v_2 \nabla^4 \phi,
\]

(\text{B.8b})

\[
\rho^2 = v_1^2 (L \cdot \nabla \phi)^2, \quad \rho^3 = 0,
\]

(\text{B.8c})

\[
\rho \partial_t \phi = u_1 v_1 (L \cdot \nabla \phi)^2.
\]

(\text{B.8d})


Introducing (B.7) and (B.8) in (B.3) and identifying the coefficients of each term we obtain (notice that \((1 + i\alpha\gamma + i\nu)/(1 + \alpha^2) = (1 + i\beta)\) in (B.3))

\[
iD_a u_1 = -2D_a^2 (1 + i\beta) v_1 - 2D_a (1 + i\alpha)
\]

(\text{B.9a})

\[
iD_a u_2 = -2D_a^2 (1 + i\beta) v_2 + iD_a (1 + i\alpha)
\]

(\text{B.9b})

\[
iD_a u_3 = -2D_a^2 (1 + i\beta) v_3 - D_a (1 + i\alpha)
\]

(\text{B.9c})

\[
u_1 v_1 + iD_a u_4 = -2D_a^2 (1 + i\beta) v_4 + 2i(1 + i\alpha) v_1
\]

(\text{B.9d})

\[
iD_a u_5 + iu_1 v_1 = -(1 + i\beta)(3D_a v_1^2 + 2D_a v_5) - 2(1 + i\alpha) v_1
\]

(\text{B.9e})

\[
u_2 v_2 + iD_a u_6 = -2D_a^2 (1 + i\beta) v_6 + (1 + i\alpha) v_2
\]

(\text{B.9f})

\[
u_2 v_1 + iD_a u_7 = -2D_a^2 (1 + i\beta) v_7 + 2i(1 + i\alpha) v_2
\]

(\text{B.9g})

\[
u_1 v_2 + iD_a u_8 = -2D_a^2 (1 + i\beta) v_8 + (1 + i\alpha) v_1
\]

(\text{B.9h})

\[
(u_4 v_1 + u_4 v_4) + iD_a u_9 = -2D_a^2 (1 + i\beta) v_9 + 2i(1 + i\alpha) v_4
\]

(\text{B.9i})

From the first three equations we can determine

\[
u_1 = 2(\beta - \alpha), \quad v_1 = -1/D_a,
\]

(\text{B.10a})

\[
u_2 = 1 + \alpha \beta, \quad v_2 = -\alpha/2D_a,
\]

(\text{B.10b})

\[
u_3 = \beta - \alpha, \quad v_3 = -1/2D_a.
\]

(\text{B.10c})
If we use the value of (B.10a)-(B.10c) we can calculate from (B.9d)-(B.9i)

\[ u_4 = -2(1 + \beta^2)/D_\infty^2, \quad v_4 = \beta/D_\infty^3, \]  
\[ u_5 = 0, \quad v_5 = -1/2 D_\infty^3, \]  
\[ u_6 = -\alpha^2(1 + \beta^2)/2 D_\infty^2, \quad v_6 = \alpha^2\beta/4 D_\infty^3, \]  
\[ u_7 = -(\alpha + \beta)(1 + \alpha \beta)/D_\infty^2, \quad v_7 = [\alpha \beta + (1 + \alpha^2)]/2 D_\infty^3, \]  
\[ u_8 = (\beta - \alpha)(1 - \alpha \beta)/D_\infty^2, \quad v_8 = [\alpha \beta - (1 + \alpha^2)]/2 D_\infty^3, \]  
\[ u_9 = 4\beta(1 + \beta^2)/D_\infty^4, \quad v_9 = -(1 + 2\beta^2)/D_\infty^5. \]  

(B.10d)  
(B.10e)  
(B.10f)  
(B.10g)  
(B.10h)  
(B.10i)

From (B.10) we obtain for the phase equation

\[
\partial_t \phi = 2\nu (L \cdot \nabla) \phi + \gamma \nabla^2 \phi + \nu(\nabla \phi)^2 - \frac{2(1 + \beta^2)}{D_\infty^2} (L \cdot \nabla)^2 \phi - \frac{\alpha^2(1 + \beta^2)}{2 D_\infty^2} \nabla^4 \phi
\]

\[
- \frac{(\alpha + \beta)(1 + \alpha \beta)}{D_\infty^2} (L \cdot \nabla) \nabla^2 \phi + \frac{(\beta - \alpha)(1 - \alpha \beta)}{D_\infty^2} \nabla^2 (L \cdot \nabla) \phi + \frac{4\beta(1 + \beta^2)}{D_\infty^4} (L \cdot \nabla)^3 \phi.
\]

(B.11)

Some consequences of this equation are:

(i) If we study the stability of a perfect pattern, the factor \(\omega t + m \varphi - S(r)\) in (B.2) will be changed to \(\omega t\) and \(D_\infty = \sqrt{\mu}\). In the phase equation we have \(L = 0\), then

\[
\partial_t \phi = (1 + \alpha \beta) \nabla^2 \phi + (\beta - \alpha) (\nabla \phi)^2 - \frac{\alpha^2(1 + \beta^2)}{2\mu} \nabla^4 \phi,
\]

(B.12)

which is the Kuramoto–Sivashinsky equation.

(ii) If we study the stability of a periodic pattern, \(\omega t + m \varphi - S(r)\) in (B.2) will be changed to \(\omega t - k \cdot r\) and \(D_\infty = \sqrt{\mu - k^2}\). In the phase equation we have \(L = -k\), and then

\[
\partial_t \phi = -2\nu (k \cdot \nabla) \phi + \gamma \nabla^2 \phi + \nu(\nabla \phi)^2 - \frac{2(1 + \beta^2)}{D_\infty^2} (k \cdot \nabla)^2 \phi - \frac{\alpha^2(1 + \beta^2)}{2 D_\infty^2} \nabla^4 \phi
\]

\[+ \frac{2\alpha(1 + \beta^2)}{D_\infty^2} (k \cdot \nabla) \nabla^2 \phi - \frac{4\beta(1 + \beta^2)}{D_\infty^4} (k \cdot \nabla)^3 \phi.
\]

(B.13)

A mode with wave vector \(p\) will be unstable if

\[ I(p) = \gamma p^2 - \frac{2(1 + \beta^2)}{D^2} (k \cdot p)^2 < 0. \]

(B.14)

The limit case is \((k \cdot p)^2 = k^2 p^2\), i.e. \(k\) and \(p\) are parallel or antiparallel. We see then that the periodic pattern will be unstable if

\[ I(k^2) = \gamma - \frac{2(1 + \beta^2)}{\mu - k^2} k^2 < 0. \]

(B.15)
Putting \( z = r \cdot \hat{k} \) and considering perturbation \( \phi \) depending only on \( z \) one has

\[
\begin{align*}
\partial_z \phi &= -2 \nu k \partial_z \phi + \left( \gamma - \frac{2(1 + \beta^2) k^2}{\mu - k^2} \right) \partial_{zz} \phi + \nu (\partial_z \phi)^2 \\
&\quad + \frac{2(1 + \beta^2) (\alpha D^2 - 2 \beta k^2) k}{D^2} \partial_{zzz} \phi + \mathcal{O}(\partial_{zzzz} \phi).
\end{align*}
\]

(B.16)

For the real Ginzburg–Landau equation (\( \alpha = \beta = 0 \)) condition (B.15) reduces to the inequality for the Eckhaus instability

\[
\frac{\mu - 3k^2}{\mu - k^2} < 0.
\]

(B.17)

(iii) For the phase stability of the spiral solution in the external region the equation is (B.11), which due to the form of \( L \) (see (2.39)) is

\[
\partial_\phi \phi = -2q \nu \frac{\partial \phi}{\partial r} + l(q^2) \frac{\partial^2 \phi}{\partial r^2} + \nu \left( \frac{\partial \phi}{\partial r} \right)^2 + \mathcal{O}\left( \frac{\partial^3 \phi}{\partial r^3} \right).
\]

(B.18)

This equation is the same as (B.16) but in the radial direction because of the fact that the spherical wave of the spiral solution (see (B.2) where \( S(r) \to qr, \ r \to \infty \)) can be considered as a plane wave in the external region. For the spiral wave centered in \( r_k \) the phase equation in the external region reduces to (2.39).

Appendix C

It can be seen from our calculations of the dynamical equations for the spirals (2.25) or for the vortices (see after (A.23)) that the only relevant term in the Ginzburg–Landau equation is the gradient term. On the other hand spiral solutions appear in a great variety of equations [6–8]. A special class of these, which are a generalization of the Ginzburg–Landau equation, are the \( \lambda-\omega \) systems:

\[
\partial_t \Lambda = (a + ib) \left[ \Lambda(|A|) + i \Omega(|A|) \right] A + \nabla^2 A,
\]

(C.1)

where \( \Lambda \) and \( \Omega \) are real functions of \( |A| \) with no singularities in \( |A| = 0 \), \( a \) and \( b \) two real numbers. The defects have here the same form (2.3) [7]:

\[
A(r, \varphi, t) = D(r) \exp\{i[\omega t + m \varphi - S(r)]\},
\]

(C.2)

with the boundary conditions (2.5)

\[
D(0) = 0, \quad S_r(0) = 0, \quad \text{and} \quad S_r(r \to \infty) = q
\]

(C.3)

we have that, as in the Ginzburg–Landau case, the solution is fixed. Introducing (C.2) in (C.1) we obtain

\[
\Delta^{-1} i \omega D = \left[ \Lambda(D) + i \Omega(D) \right] D + \nabla^2 D - D \left( \frac{m^2}{r^2} + S_r^2 \right) - 2i D S_r - i D \nabla^2 S,
\]

(C.4)
where \( \Delta = (a + ib) \). Imposing the asymptotic condition \( (S_r \to q, D \to D_w) \) we find

\[
\Delta^{-1} i \omega = \Lambda(D_w) + i \Omega(D_w) - q^2,
\]

(C.5)

and we have two equations which determine \( \omega \) and \( D_w \). In the following we assume that \( D_w \) exists and is different from zero.

Introducing (C.5) in (C.4) we obtain two coupled differential equations:

\[
\frac{\nabla^2 D}{D} = \left( S_r^2 + \frac{m^2}{r^2} - q^2 \right) - \left[ \Lambda(D_w) - \Lambda(D) \right] = 0,
\]

(C.6a)

\[
\frac{\nabla^2 S + 2 \frac{D_r S_r}{D}}{D} = - \left[ \Omega(D_w) - \Omega(D) \right].
\]

(C.6b)

The asymptotic behavior is, for one-armed spiral waves \( |m| = 1 \)

\[
D(r) \to \lambda r, \quad r \to 0; \quad \frac{D(r)}{\Omega(D_w)} \to \frac{1}{r}, \quad r \to \infty,
\]

(C.7a)

\[
S_r \text{ known}, \quad r \to 0; \quad S_r(r) \to q + \frac{\Lambda'(D_w)}{2\Omega(D_w)} \frac{1}{r}, \quad r \to \infty.
\]

(C.7b)

Here the prime denotes the derivative with respect to the argument. From (C.6b) we see that \( S_r = 0 \) if \( \Omega = 0 \); this is the vortex solution.

Now we proceed to describe a state of (C.1) representing a diluted gas of spirals like the one of section 2. The ansatz is given again by (2.9) and (2.10). Introducing the ansatz (2.9) in eq. (C.1), using the value of \( \omega \) obtained from (C.5), neglecting the term \( \dot{\rho} w \) and keeping only linear terms in \( w \) we obtain

\[
\Delta^{-1} \left( \delta_r R^0 + i R^0 \delta_\theta \theta^0 \right) - h(R^0) R^0 - \nabla^2 R^0 + R^0 \left[ (\nabla \theta^0)^2 - q^2 \right] - 2i \nabla R^0 \cdot \nabla \theta^0 - i R^0 \nabla^2 \theta^0 = h(R^0) w + R^0 \left[ \Lambda'(R^0) + i \Omega(R^0) \right] \text{Re} w + \left\{ \nabla^2 - \left( (\nabla \theta^0)^2 - q^2 \right) + 2i \nabla \theta^0 \cdot \nabla + i \nabla^2 \theta^0 \right\} w,
\]

(C.8)

where

\[
h(R^0) = -\Lambda(D_w) + \Lambda(R^0) + i \left[ \Omega(D_w) - \Omega(R^0) \right].
\]

We then obtain a linear equation for \( w \) as in section 2: \( \mathcal{L}^w = I \). The solvability condition is that \( I \) is orthogonal to the kernel of \( \mathcal{L}^\dagger \). We look for a first class of vectors in \( \text{ker} \mathcal{L}^\dagger \) which vanish outside the core of the \( k \)th spiral (see appendix A); then \( \mathcal{L}^\dagger \) takes the form

\[
\mathcal{L}^\dagger \approx \nabla^2 - \frac{m^2}{\rho_k^2} + 2i \frac{m_k}{\rho_k} \hat{\phi}_k \cdot \nabla,
\]

(C.10)

(note that (C.10) is the analogue of (2.18)). Here we have used scalar product

\[
\langle f, g \rangle = \int d\vec{r} \hat{f} \hat{g}.
\]
We now determine elements $Z^{(n)}_k$ in $\ker\mathcal{E}$ of the form

$$ Z^{(n)}_k(\rho_k) = c_n P^{(n+m_k)}_k e^{im_k\varphi_k}. \quad (C.11) $$

The scalar product of these elements with $I$ is different from zero only for $n = \pm 1$. We keep only the dominant term of $Z^{(n)}_k$, i.e. $n = -m_k$. Then

$$ Z_k(\rho_k) = e^{-im_k\varphi_k} + \mathcal{E}(\rho_k^2). \quad (C.12) $$

(Note that this kernel element is the analogue of the two kernel elements (2.19).)

The solvability condition gives us

$$ \int_{V_c(r_k)} d\rho_k e^{im_k\varphi_k} \left[ \Delta^{-1} \left( \partial_\alpha R^0 + i R^0 \partial_\beta \Theta^0 \right) + R^0 \left( \nabla \Theta^0 \right)^2 - 2i \nabla R^0 \cdot \nabla \Theta^0 \right] = 0, \quad (C.13) $$

where we have eliminated the terms without angular dependence $\varphi_k$. Defining $L_k$ as in section 2 we obtain a new equation which replaces (2.21):

$$ \int d\rho_k e^{im_k\varphi_k} \left\{ -\Delta^{-1} \left[ D'(\rho_k) \hat{\rho}_k + iD(\rho_k) L_k \right] \cdot \hat{\varphi}_k - 2i \left[ D'(\rho_k) \hat{\rho}_k + iD(\rho_k) L_k \right] \cdot \nabla \Phi^{(k)} \right\} = 0 $$

$$ \quad (C.14) $$

after using

$$ \int_{0}^{2\pi} d\varphi_k e^{im_k\varphi_k} \hat{\rho}_k = \pi(1, im_k) = \pi j_\rho, $$

$$ \int_{0}^{2\pi} d\varphi_k e^{im_k\varphi_k} \hat{\varphi}_k = \pi(-im_k, 1) = \pi j_\varphi = -im_k \pi j_\rho, $$

and evaluating $\nabla \Phi^{(k)}(r_k)$ in $r_k$. We now proceed to integrate in $\rho_k$ using the integrals in section 2, then

$$(2j_\rho + 2iKj_\rho) \cdot \hat{r}_k = -2i \Delta (2j_\rho + 2iKj_\rho) \cdot \nabla \Phi^{(k)}, \quad (C.15) $$

and the constant $K$ cancels to give

$$ j_\rho \cdot \hat{r}_k = -2i \Delta j_\rho \cdot \nabla \Phi^{(k)}(r_k). \quad (C.16) $$

Using the definition of $j_\rho$ and $\Delta$, we see that the complex expression (C.16) is separated in two real expressions, which are the equations of motion [3]

$$ \frac{d\hat{r}_k}{dt} = -2am_k \hat{z} \wedge \frac{\partial \Phi^{(k)}(r_k)}{\partial r_k} + 2b \frac{\partial \Phi^{(k)}(r_k)}{\partial r_k}. \quad (C.17) $$

This is a generalization of eq. (2.25) and can be interpreted saying that the interaction is a linear combination of the results obtained by Kawasaki (1983) [7] and Fetter (1966) [8] realized for the cases $b = 0$ and $a = 0$ respectively.
Developing the expression of $\Phi^{(k)}$ (2.16c), we obtain

$$\frac{d r_k}{dt} = 2 \left( a \sum_{i \neq k} \frac{m_i m_k}{r_{ik}} \hat{r}_{ik} - b \sum_{i \neq k} \tilde{S}_r(r_{ik}) \hat{r}_{ik} + b \nabla \Psi(r_k) \right) + 2 m_k \hat{z} \wedge \left( b \sum_{i \neq k} \frac{m_i m_k}{r_{ik}} \hat{r}_{ik} + a \sum_{i \neq k} \tilde{S}_r(r_{ik}) \hat{r}_{ik} - a \nabla \Psi(r_k) \right),$$

(C.18)

where $\tilde{S}_r$ is given by (C.7b).

For the phase dynamics we proceed as in section 2 to obtain the new element of the kernel. In order to do this we use the matrix expression associated to the linear operator $\mathcal{L}$ separating the real part and the imaginary part of eq. (C.8). One obtains

$$\chi_{ii}^{(k)} = \begin{pmatrix} -\Omega'(D_\omega) \\ \Lambda'(D_\omega) \end{pmatrix},$$

(C.19)

and the phase equation is

$$\frac{a \Lambda' - b \Omega'}{a^2 + b^2} \partial_t \Psi = \Lambda' \nabla^2 \Psi + \Omega'(\nabla \Psi)^2 + 2 \Omega'(L_k + \Sigma i_j) \cdot \nabla \Psi + 2 \Omega'L_k \cdot \Sigma i_j + \ldots,$$

(C.20)

where $\Omega'$ and $\Lambda'$ are evaluated in $D = D_\omega$, $\Omega' = \Omega'(D_\omega)$ and $\Lambda' = \Lambda'(D_\omega)$.

We now discuss the problem of superfluidity. We consider the condensate wavefunction $\psi$ that satisfies the Ginzburg-Gross-Pitaevskii equation [17, 18]

$$i \hbar \partial_t \psi(r, t) = -\frac{\hbar^2}{2m} \nabla^2 \psi - \mu \psi + U_0 |\psi|^2 \psi,$$

(C.21)

where $2\pi \hbar$ is the Planck constant, $m$ the He$^4$ atomic mass, $\mu$ the chemical potential and $U_0$ measures the interatomic interaction. The defects here are vortices (appendix A). We now look for a solution $\psi$ representing a diluted gas of vortices. We identify (C.21) with (C.1), i.e. we have $a = \Omega = 0$, $b = \hbar/2m$ and

$$A(|\psi|) = \frac{2m}{\hbar^2} (\mu - U_0 |\psi|^2).$$

The vortex dynamics is obtained from (C.18) (note that $S_r = 0$ since $\Omega = 0$)

$$\frac{d r_k}{dt} = 2 b \sum_{i \neq k} \frac{m_i}{r_{ik}} \hat{r}_{ik} + 2 b \nabla \Psi(r_k).$$

(C.22)

From (B.20) we obtain the phase equation $\nabla^2 \Psi = 0$. The nonsingular solution of this equation is $\Psi = 0$, and (C.22) reduces to the usual vortex dynamics, normally obtained by arguments of hydrodynamics [8, 17]:

$$\frac{d r_k}{dt} = \frac{1}{2\pi} \sum_{i \neq k} \frac{I_i}{|r_k - r_i|} \hat{z} \wedge \frac{(r_k - r_i)}{|r_k - r_i|},$$

(C.23)
where

$$\Gamma_i = (\hbar/m)m_i$$

is the circulation, which satisfies the Feynman–Onsager rule of quantization of the vortex circulation [18].

Appendix D

The case of spirals with arbitrary topological charge ($|m|$ arbitrary) requires some small changes. We follow appendix C and make there the following changes:

(i) The asymptotic behavior of an isolated spiral solution of charge $m$ is

$$D(r) \to \lambda_m r^{|m|}, \quad r \to 0; \quad D(r) \to D_\infty + \frac{q_m}{\Omega'(D_\infty)} \frac{1}{r}, \quad r \to \infty,$$

$$S_r \text{ known, } \quad r \to 0; \quad S_s(r) \to q_m + \frac{\Lambda'(D_\infty)}{2\Omega'(D_\infty)} \frac{1}{r}, \quad r \to \infty.$$  (D.1a)

We see here that the value of $q$ depends on $m$ and consequently $\omega$ and $D_\infty$ depend also on $m$. We then have to change slightly the ansatz since now $\omega$ and $D_\infty$ take different values in the different regions $D_k$ around the $k$th (see section 2) spiral. Taking into account that the relevant property the ansatz $A = (R^0 + w) \exp(i\Theta^0)$ has to fulfill is that around each point $r_k$, the location of the center of the $k$th spiral, the behavior must be locally that of a spiral, we put

$$R_k = D_0(r_k) + \ldots,$$

$$\Theta^0 = \omega(q_k) + m_k \varphi_k - S_r(p_k) + \Phi^{(k)}(r, t),$$

$$\Phi^{(k)}(r, t) = \sum_{i \neq k} \left( m_i \varphi_i - S_s(p_i) \right) + \Psi(r, t),$$

where $\tilde{S}(p_i) = S(p_i) - (q_m p_i)$ is given by (D.1b).

(ii) The operator $\mathcal{L}^1$ is the same as (C.10) and the element $Z^{(n)}_k$ is the same as (C.11), but now the dominant term is

$$Z_k(p_k) = \rho_{k|m_k^{-1}|} e^{-i \bar{m}_k \varphi_k} + \mathcal{O}(\rho_{k|m_k^{-1}+1}),$$

where we have defined

$$\bar{m}_k = m_k/|m_k|.$$  (D.3)

(iii) The solvability condition gives now instead of (C.14) the equation

$$\int d\rho_k \rho_{k|m_k^{-1}|} e^{i \bar{m}_k \varphi_k} \left[ D'(\rho_k) \dot{\rho}_k + i D(\rho_k) L_k \right] \cdot\left( -\Delta^{-1} \dot{r}_k - 2i \nabla \Phi^{(k)} \right) = 0.$$  (D.4)
(iv) We define now $j_p$ as

$$
\int_0^{2\pi} d\Psi_k e^{i\hat{\Psi}_k} \hat{\rho}_k = \pi (1, i\hat{\Psi}_k) = \pi j_p,
$$

and doing the integrals (D.4) gives the final equation

$$
j_p \cdot \dot{r}_k = -2i\Delta j_p \cdot \nabla \Phi^{(k)}(r_k),
$$

which in the usual vector form is

$$
\frac{dr_k}{dt} = -2a \frac{m_k}{|m_k|} \hat{z} \wedge \nabla \Phi^{(k)}(r_k) + 2b \nabla \Phi^{(k)}(r_k).
$$

Developing $\Phi^{(k)}$ using (D.2c) we obtain

$$
\frac{dr_k}{dt} = 2\left( a \frac{m_k}{|m_k|} \sum_{i \neq k} \frac{m_i}{r_{ik}} \hat{r}_{ik} - b \sum_{i \neq k} \bar{S}_r(r_{ik}) \hat{r}_{ik} + b \nabla \psi(r_k) \right)
+ 2\hat{z} \wedge \left( b \sum_{i \neq k} \frac{m_i}{r_{ik}} \hat{r}_{ik} + a \frac{m_k}{|m_k|} \sum_{i \neq k} \bar{S}_r(r_{ik}) \hat{r}_{ik} - a \frac{m_k}{|m_k|} \nabla \psi(r_k) \right),
$$

where $\bar{S}_r(r_{ik}) = [A(D_o)/2\Omega(D_o)](1/r_{ik})$ is given by (D.1b).

(v) The phase equation (C.20) is the same but $L_k$ now depends on $q_{mk}$ through the function $S_r(\rho_k)$. We remark that in the case of the Ginzburg–Landau equation $\bar{S}_r(r_{ik}) = (\gamma/2\nu)(1/r_{ik})$ is independent of $q_{mk}$.

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Note added in proof

It should be pointed out that the central result here is the set of coupled equations (2.25) and (2.29) for the trajectories of the spirals $\{r_j(t)\}$ and the new global phase $\psi(r, t)$ defined by (2.15). In fact in (2.25) what appears if one looks carefully at our derivation is $\langle \nabla \Phi^{(k)} \rangle$, where $\langle \rangle$ stands for spatial average in the core of the $k$th spiral, and since $\langle L_k \rangle = 0$ one can replace $\langle \nabla \Phi^{(k)} \rangle$ by $\langle \nabla \theta^0 \rangle$. We have not solved these coupled equations in the general case here and this remains the essential new step to be done. In the case $\alpha = \beta$ we can solve them (these calculations will be presented elsewhere) and the conclusions we obtain are in agreement with previous results of other authors for vortices in the real Ginzburg–Landau equation and in the nonlinear Schrödinger equation which corresponds to the limit $\alpha = \beta \to \infty$. 


References